Exercises CFT-course fall 2013, set 10.

1. Constraints on c and h_i from the fusion rules.

   Different conformal field theories can have the same fusion rules. However, for a given set of fusion rules, the possible values of c and h_i are restricted by modular invariance.

   We will consider a theory with three fields $\mathbf{1}$, σ and ψ, with the fusion rules given by $\sigma \times \sigma = \mathbf{1} + \psi$, $\sigma \times \psi = \sigma$ and $\psi \times \psi = \mathbf{1}$. Thus the fusion matrices are (the fields are ordered as $\mathbf{1}, \sigma, \psi$)

$$N_{\mathbf{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad N_\psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   These fusion rules are diagonalized by

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} .$$

   Show that the general relation $(ST)^3 = C$, where $C = S^2$ is the conjugation matrix (which satisfies $C^2 = 1$) and $T_{i,j} = e^{2\pi i (h_i - c/24)} \delta_{i,j}$, gives rise to the constraints

$$h_\psi = \frac{1}{2} \bmod 1 \qquad\qquad h_\sigma - c/8 = 0 \bmod 1 .$$

2. The pentagon equation.

   We will consider the so-called Fibonacci fusion rules, consisting of two fields, the identity $\mathbf{1}$ and a field τ, such that the only non-trivial fusion rule is $\tau \times \tau = \mathbf{1} + \tau$. These fusion rules are, for instance, realized by the non-unitary minimal model $\mathcal{M}(2,5)$.

   This exercise amounts to solving the pentagon equations for these fusion rules. To streamline the process, you may assume the following fact about the F-symbols. By making use of the gauge degrees of freedom, one can set $\left(F_d^{a,b,c}\right)_{e,f} = 1$, if any of a, b, c equals the vacuum $\mathbf{1}$. Note that this does not completely fix the gauge for the Fibonacci fusion rules.

   Find the remaining F-symbols, by considering the pentagon equations where all incoming labels are set to τ. You do not have to check that your solution satisfies the other pentagon equations.

3. The hexagon equation.

   Use the unitary solution of the previous exercise to solve the hexagon equations. Use the gauge for which $R_c^{a,b} = 1$ if $a = \mathbf{1}$ or $b = \mathbf{1}$. Again, consider the equations with all incoming lines labeled by τ.