Exercises CFT-course fall 2013, set 9.

Due on december 20th, 2013.

1. Character formula's for the Ising model.

a. Calculate the character of the vacuum and ψ sector by calculating the partition function of states of the form

$$\psi_{-(n-1)/2-p_n} \cdots \psi_{-3/2-p_2}\psi_{-1/2-p_1}|0\rangle \ ,$$

where $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

b. Repeat exercise a. for the σ sector, by considering

$$\psi_{-(n-1)-p_n} \cdots \psi_{-1-p_2}\psi_{-p_1}|\sigma\rangle \ ,$$

again with $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

Answer:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} = \prod_{n=1}^{\infty}(1+q^n) = \sum_{n\geq 0 \text{ even}} \frac{q^{n(n-1)/2}}{(q)_n} = \sum_{n\geq 1 \text{ odd}} \frac{q^{n(n-1)/2}}{(q)_n} \ . \qquad (1)$$

c. Obtain the last two equalities in (1), by making use of an identity due to Cauchy (which you don't have to prove):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}x^n}{(q)_n} = \prod_{n=0}^{\infty}(1+xq^n) \ ,$$

and considering the role of the zero mode ψ_0.

2. Modular transformation properties of the Ising characters.

The following definitions and identities are given ($q = e^{2\pi i\tau}$):

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2 \qquad \eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2$$

a. Show, by making use of the results of exercise 1., that the characters of the Ising model are given by

$$\chi_0 = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/16} = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$$

b. Calculate the transformation properties of the Ising characters under the transformation $\tau \to -1/\tau$, by making use of the Poisson resummation formula:

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{(b+2\pi i k)^2/(4\pi a)}$$

3. CFT description of the transverse-field Ising model.

In exercise 1. of the first set of exercises, we considered the transverse-field Ising model in one dimension. The hamiltonian in terms of the Pauli matrices σ was given by

$$H_I = -J \sum_i \left(g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \ .$$

You showed that the hamiltonian can be diagonalized, with as the final result

$$H_I = \sum_k \varepsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \qquad\qquad \varepsilon_k = 2(1 + g^2 - 2g\cos(ak))^{\frac{1}{2}}$$

Here, the $\gamma_k^{(\dagger)}$'s are fermion creation and annihilation operators with momentum k. For F odd, the momenta take the values $k = 0, 1, \ldots, L-1$, while or F even, the allowed momenta are $k = 1/2, 3/2, \ldots, L - 1/2$, because of the periodic boundary conditions we imposed.

In this exercise, we will consider the critical behaviour at $g = 1$, for $J = 1$. Although the model has been solved exactly, it is not so easy to extract the critical behaviour analytically, because of the sum over k, which one has to evaluate to obtain the energies. Therefor, you can use some computer program to evaluate the energies for finite system size.

a. Evaluate the the ground state energy (at $g = J = 1$) numerically for finite system sizes L, (up to $L = 500$ should not be a problem) and fit the results to the (general) finite size scaling formula

$$E_i = E_0 L + \frac{2\pi v}{L} \left(-\frac{c}{12} + h_l + h_r \right) \ ,$$

where E_0 is a constant energy per spin, v a velocity, c the central charge, and h_l and h_r the left and right scaling dimensions of scaling fields.

b. Show numerically that the system is critical, by obtaining the finite size gap between the ground and first excited states.

c. Obtain the other constants in the scaling formula and dimensions of the primary fields by considering various low-lying states in the spectrum.

d. (optional) Obtain the full spectrum for a small system (say $L = 10$), and plot the states as a function of momentum.