Exercises CFT-course fall 2013, set 6.

Due on november 15, 2013.

1. A little on partitions.

   Let $p_d(n)$ be the number of partitions of $n \geq 0$ into *distinct* parts. Thus, f.i. $p_d(5) = 3$, namely $5 = 1 + 4$, $5 = 2 + 3$, $5 = 5$.

   Let $p_o(n)$ be the number of partitions of $n \geq 0$ into *odd* parts. Thus, f.i. $p_o(5) = 3$, namely $5 = 1 + 1 + 1 + 1 + 1$, $5 = 1 + 1 + 3$, $5 = 5$.

   Show that $p_d(n) = p_o(n)$ by obtaining their generating functions, and showing that they are equal.

2. Highest weight condition.

   Show that the highest weight condition, namely $L_n|\chi\rangle = 0$ for all $n > 0$, is implied by the conditions $L_1|\chi\rangle = L_2|\chi\rangle = 0$.

3. Null vector at level 3.

   Find an explicit expression for the null vector at level 3. Answer:

   $$|\chi_{1,3}\rangle = \big(L_{-3} - \frac{2}{h}L_{-2}L_{-1} + \frac{1}{h(h+1)}(L_{-1})^3\big)|h\rangle$$

   Determine the corresponding central charge c as a function of h.

4. Unitarity of $SU(2)$ representations.

   Consider the $SU(2)$ algebra

   $$[J_+, J_-] = 2J_0 \qquad [J_0, J_\pm] = \pm J_\pm$$

   and let $|j\rangle$ be a highest weight, $J_+|j\rangle = 0$, with J_0 eigenvalue j. Show that one can only construct unitary highest weight representations for j either a non-negative integer of a positive half-integer.