Exercises CFT-course fall 2013, set 1.

Due on Friday, september 20th.

The transverse-field Ising model.

The hamiltonian of the transverse-field Ising model in one dimension reads, in terms of the Pauli matrices σ,

$$H = -J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + h \sigma_i^z \right) .$$

This model can be solved by means of a Jordan-Wigner transformation, which transforms the spin operators into fermionic ones. For simplicity, we will set $J = 1$ throughout the exercise. To transform the spin degrees of freedom into (spin-less) fermions, we let a spin-up at site i correspond to an empty site. Conversely, a spin-down corresponds to a site occupied by a fermion. If we consider a single site i, the spin raising operator σ_i^+ corresponds to the fermionic annihilation operator c_i. Conversely, $\sigma_i^- = c_i^\dagger$. These operators indeed satisfy the fermionic anti-commutation relations (with $i = j$)

$$\{c_i, c_j^\dagger\} = \delta_{i,j} \qquad\qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 .$$

However, the spin operators on different sites commute, so the construction above does not work for $i \neq j$.

Jordan and Wigner solved this problem, by introducing

$$\sigma_i^z = 1 - 2c_i^\dagger c_i \qquad \sigma_i^+ = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j)\Big) c_i \qquad \sigma_i^- = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j)\Big) c_i^\dagger .$$

The inverse is

$$c_i = \Big(\prod_{j<i} \sigma_j^z\Big) \sigma_i^+ \qquad\qquad c_i^\dagger = \Big(\prod_{j<i} \sigma_j^z\Big) \sigma_i^- .$$

a. Show that the σ operators defined above satisfy the correct commutation relations, by using the fermion anti-commutation relations given above.

b. We will assume that the chain contains L sites, and is closed, so that the sites $i = L$ and $i = 0$ are identified (i.e., we assume $\sigma_{i+L}^\alpha = \sigma_i^\alpha$). Show that after the Jordan-Wigner transformation, the hamiltonian reads

$$\begin{aligned} H = &\sum_{i=0}^{L-1} g(2c_i^\dagger c_i - 1) + \sum_{i=0}^{L-2} \left(c_i c_{i+1} + c_i c_{i+1}^\dagger + c_{i+1} c_i^\dagger + c_{i+1}^\dagger c_i^\dagger \right) \\ &- (-1)^F \left(c_{L-1} c_0 + c_{L-1} c_0^\dagger + c_0 c_{L-1}^\dagger + c_0^\dagger c_{L-1}^\dagger \right) , \end{aligned}$$

where F is the number of fermions, which is conserved modulo two.

The boundary term can be taken into account by imposing periodic boundary conditions $c_L = c_0$ when F is odd, and anti-periodic boundary conditions $c_L = -c_0$ when F is even. With these boundary conditions, the hamiltonian is translationally invariant, namely

$$H = \sum_{i=0}^{L-1} g(2c_i^\dagger c_i - 1) + \sum_{i=0}^{L-1} \bigl(c_i c_{i+1} + c_i c_{i+1}^\dagger + c_{i+1} c_i^\dagger + c_{i+1}^\dagger c_i^\dagger\bigr) \ .$$

c. To diagonalize the hamiltonian, we first have to go to momentum space, via $c_j = \frac{1}{\sqrt{L}} \sum_k c_k e^{iakj}$, with $a = \frac{2\pi}{L}$. Show that the hamiltonian takes the following form:

$$H = \sum_k 2(g - \cos(ak)) c_k^\dagger c_k + i \sin(ak)(c_{-k} c_k + c_{-k}^\dagger c_k^\dagger) - g \ .$$

To bring the hamiltonian in diagonal form, one performs a Bogoliubov transformation to a different set of fermions:

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger \qquad\qquad \gamma_k^\dagger = u_k c_k^\dagger + i v_k c_{-k} \ ,$$

where u_k and v_k are real, and satisfy $u_k^2 + v_k^2 = 1$, $u_{-k} = u_k$ and $v_{-k} = -v_k$, ensuring that the γ_k satisfy fermion anti-commutation relations.

d. Choose the parametrization $u_k = \cos\theta_k/2$ and $v_k = \sin\theta_k/2$, and find the condition on $\tan\theta_k$, such that terms which do not conserve the number of γ fermions (like $\gamma\gamma$) are absent in the hamiltonian, expressed in terms of the γ's.

e. Finally, show that the hamiltonian takes the diagonal form

$$H = \sum_k \varepsilon_k \bigl(\gamma_k^\dagger \gamma_k - \frac{1}{2}\bigr) \qquad\qquad \varepsilon_k = 2(1 + g^2 - 2g\cos(ak))^{\frac{1}{2}}$$

Note that for F odd, the momenta take the values $k = 0, 1, \ldots, L-1$, while or F even, the allowed momenta are $k = 1/2, 3/2, \ldots, L - 1/2$, because of the boundary conditions.