Exercises CFT-course fall 2011, set 10.

Due on friday, december 23rd (well...).

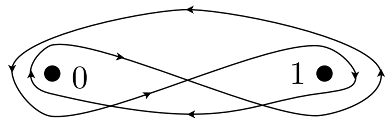
1. Integral expressions for the hypergeometric function.
   Recall the results of exercise 3, set 6.

   a. Consider the differential operator $D = z(1-z)\partial_z^2 + \big(c-(a+b+1)z\big)\partial_z - ab$. Show that the action of D on the monomial $t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$ is a total derivative with respect to t.

   b. Argue that the expression
   $$\int_{\mathcal{C}} dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$
   is a solution of the hypergeometric differential equation. What types of curves $\mathcal{C}$ can one take?

   c. Show that the hypergeometric function $F(a,b;c;z)$ can be expressed as
   $$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$
   by performing an appropriate expansion and using an integral expression for the beta function $B(x,y)$ (assume that $\Re(c) > \Re(b) > 0$ and $|z| < 1$).

   d. Let γ be the following curve in the complex plane

   Show that
   $$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{1}{(1-e^{2\pi i b})(1-e^{2\pi i(c-b)})}\int_\gamma dt t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$
   for $b, c-b \notin \mathbf{Z}$.

2. An Ising model correlation function from the Coulomb gas formalism.

   a. Write the two conformal blocks $I_1(z), I_2(z)$ for the field $\psi = \phi_{1,2}$ in the Ising model, with $(p,p') = (3,4)$, and find the values of a, b, c.

   b. The fusion rule $\psi \times \psi = 1$ indicates that there should only be one conformal block. To reconcile the results, consider the full correlation function, which takes the form (you don't have to show this)
   $$G(z,\bar{z}) \propto \frac{\sin\big(b\pi\big)\sin\big((a+b+c)\pi\big)}{\sin\big((a+c)\pi\big)}|I_1(z)| + \frac{\sin\big(a\pi\big)\sin\big(c\pi\big)}{\sin\big((a+c)\pi\big)}|I_2(z)|\ .$$