Exercises CFT-course fall 2011, set 8.

Due on december 9th, 2011.

1. Character formula's for the Ising model.

a. Calculate the character of the vacuum and ψ sector by calculating the partition function of states of the form

$$\psi_{-(n-1)/2-p_n} \cdots \psi_{-3/2-p_2} \psi_{-1/2-p_1} |0\rangle \ ,$$

where $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

b. Repeat exercise a. for the σ sector, by considering

$$\psi_{-(n-1)-p_n} \cdots \psi_{-1-p_2} \psi_{-p_1} |\sigma\rangle \ ,$$

again with $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

Answer:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} = \prod_{n=1}^{\infty}(1+q^n) = \sum_{n\geq 0 \text{ even}} \frac{q^{n(n-1)/2}}{(q)_n} = \sum_{n\geq 1 \text{ odd}} \frac{q^{n(n-1)/2}}{(q)_n} \ . \tag{1}$$

c. Obtain the last two equalities in (1), by making use of an identity due to Cauchy (which you don't have to prove):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} x^n}{(q)_n} = \prod_{n=0}^{\infty}(1+xq^n) \ ,$$

and considering the role of the zero mode ψ_0.

2. Modular transformation properties of the Ising characters.

The following definitions and identities are given ($q = e^{2\pi i \tau}$):

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2} = 2q^{1/8} \prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \qquad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$$

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2 \qquad \eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2$$

a. Show, by making use of the results of exercise 1., that the characters of the Ising model are given by

$$\chi_0 = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/16} = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$$

b. Calculate the transformation properties of the Ising characters under the transformation $\tau \to -1/\tau$, by making use of the Poisson resummation formula:

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{(b+2\pi i k)^2/(4\pi a)}$$

3. Constraints on c and h_i from the fusion rules.

Different conformal field theories can have the same fusion rules. However, for a given set of fusion rules, the possible values of c and h_i are restricted by modular invariance.

We will consider a theory with three fields $\mathbf{1}$, σ and ψ, with the fusion rules given by $\sigma \times \sigma = \mathbf{1} + \psi$, $\sigma \times \psi = \sigma$ and $\psi \times \psi = \mathbf{1}$. Thus the fusion matrices are (the fields are ordered as $\mathbf{1}, \sigma, \psi$)

$$N_{\mathbf{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad N_\psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

These fusion rules are diagonalized by

$$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} .$$

Show that the general relation $(ST)^3 = C$, where $C = S^2$ is the conjugation matrix (which satisfies $C^2 = 1$), and $T_{i,j} = e^{2\pi i (h_i - c/24)} \delta_{i,j}$ gives rise to the constraints

$$h_\psi = \frac{1}{2} \bmod 1 \qquad\qquad h_\sigma - c/8 = 0 \bmod 1 .$$