Exercises CFT-course fall 2008, set 4 & 5.

Due on november 11, 2011.

1. Mode expansion of primary fields.

   In analogy to the stress-energy tensor, we define a mode expansion of (holomorphic) primary fields $\phi(z)$ as

$$\phi(z) = \sum_{n \in \mathbb{Z}-h} \phi_n z^{-n-h} \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{h+n-1} \phi(z) \ ,$$

   where ϕ_{-n} has weight n.

a. Show that regularity requires $\phi_n|0\rangle = 0$ for $n \geq -h+1$.

b. Calculate the commutator $[L_n, \phi_m]$ by first calculating $[L_n, \phi(w)]$. Show that $L_0 \phi_{-h}|0\rangle = h|h\rangle$, as expected.

2. The OPE of descendent fields.

   Show that the OPE of $T(z)$ with $\phi^{(-1)}(w) = \partial\phi(w)$, where $\phi(w)$ is a primary field of dimension h, takes the form

$$T(z)\partial\phi(w) = \frac{2h\phi(w)}{(z-w)^3} + \frac{(h+1)\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} + \text{reg} \ .$$

3. Highest weight condition.

   Show that the highest weight condition, namely $L_n|\chi\rangle = 0$ for all $n > 0$, is implied by the conditions $L_1|\chi\rangle = L_2|\chi\rangle = 0$.

4. Null vector at level 3.

   Find the explicit expression for the null vector at level 3. Answer:

$$|\chi_{1,3}\rangle = \big(L_{-3} - \frac{2}{h} L_{-2} L_{-1} + \frac{1}{h(h+1)} (L_{-1})^3\big)|h\rangle$$

   Determine the corresponding central charge c as a function of h.

5. A little on partitions.

   Let $p_d(n)$ be the number of partitions of $n \geq 0$ into *distinct* parts. Thus, f.i. $p_d(5) = 3$, namely $5 = 1+4$, $5 = 2+3$, $5 = 5$.

   Let $p_o(n)$ be the number of partitions of $n \geq 0$ into *odd* parts. Thus, f.i. $p_o(5) = 3$, namely $5 = 1+1+1+1+1$, $5 = 1+1+3$, $5 = 5$.

   Show that $p_d(n) = p_o(n)$ by obtaining their generating functions, and showing that they are equal.

6. Unitarity of $SU(2)$ representations.

   Consider the $SU(2)$ algebra

$$[J_+, J_-] = 2J_0 \qquad [J_0, J_\pm] = \pm J_\pm$$

   and let $|j\rangle$ be a highest weight, $J_+|j\rangle = 0$, with J_0 eigenvalue j. Show that one can only construct unitary highest weight representations for j either a non-negative integer of a positive half-integer.