Exercises CFT-course fall 2011, set 3.

Due on friday, october 14, 2011.

1. The two point function on the cylinder.

a. Calculate the two point function of a primary operator $\phi(z,\bar{z})$ with scaling dimensions $(h,\bar{h})$ on the cylinder, by using the conformal map $w = \frac{L \log z}{2\pi}$. Result:

$$\langle \phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle = \big(\frac{2\pi}{L}\big)^{2h+2\bar{h}}\big(2\sinh\big(\pi(w_1-w_2)/L\big)\big)^{-2h}\big(2\sinh\big(\pi(\bar{w}_1-\bar{w}_2)/L\big)\big)^{-2\bar{h}}$$

b. Let $w = t + iu$. Analyze the result of a. in the limits $t_1 - t_2 \ll L$ and $t_1 - t_2 \gg L$. For simplicity, you may assume that $h = \bar{h}$.

2. The massless free boson ϕ has the action

$$S = \frac{g}{2}\int dx^2 \partial_\mu \phi \partial^\mu \phi \ .$$

a. Determine the propagator of the free massless boson theory:

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = \frac{-1}{4\pi g}\big(\ln(z-w)+\ln(\bar{z}-\bar{w}\big) + \text{const.}$$

b. Determine the form of the stress-energy tensor, namely $T(z) = -2\pi g :\partial\phi(z)\partial\phi(z):$.

c. Calculate the OPE $T(z)T(w)$.

d. Show that the vertex operator $V =: e^{i\alpha\phi} : (z)$ is a primary field of dimension $h_\alpha = \frac{\alpha^2}{8\pi g}$.

3. The massless free fermion.

The action of the massless free fermion can be written as

$$S = g\int dx^2(\overline{\psi}\partial\overline{\psi} + \psi\overline{\partial}\psi) \ ,$$

where ψ and $\overline{\psi}$ are the two components of the spinor Ψ. It is given that the equations of motion are $\partial\overline{\psi} = \overline{\partial}\psi = 0$, while the propagator takes the form $\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle = \frac{1}{2\pi g}\frac{1}{z-w}$. The stress-energy tensor can be written as $T(z) = -\pi g : \psi(z)\partial\psi(z):$.

a. Show that $\psi(z)$ is a primary field.

b. Calculate the OPE $T(z)T(w)$.

4. The Virasoro algebra.

Show that the Virasoro algebra has the following form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ ,$$

where $L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z)$.