Exercises CFT-course fall 2011, set 1.

Due on Friday, september 23rd.

1. The transverse-field Ising model.

   The hamiltonian of the transverse-field Ising model in one dimension reads, in terms of the Pauli matrices σ,

   $$H_I = -J \sum_i \left(g \sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right) \ .$$

   This model can be solved by means of a Jordan-Wigner transformation, which transforms the spin operators into fermionic ones. For simplicity, we will set $J = 1$ throughout the exercise.

   To transform the spin degrees of freedom into (spin-less) fermions, we let a spin-up at site i correspond to an empty site. Conversely, a spin-down corresponds to a site occupied by a fermion. If we consider a single site i, the spin raising operator σ_i^+ corresponds to the fermionic annihilation operator c_i. Conversely, $\sigma_i^- = c_i^\dagger$. These operators indeed satisfy the fermionic anti-commutation relations (with $i = j$)

   $$\{c_i, c_j^\dagger\} = \delta_{i,j} \qquad\qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \ .$$

   However, the spin operators on different sites commute, so the construction above does not work for $i \neq j$.

   Jordan and Wigner solved this problem, by introducing

   $$\sigma_i^z = 1 - 2c_i^\dagger c_i \qquad \sigma_i^+ = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j)\Big) c_i \qquad \sigma_i^- = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j)\Big) c_i^\dagger \ .$$

   The inverse is

   $$c_i = \Big(\prod_{j<i} \sigma_j^z\Big) \sigma_i^+ \qquad\qquad c_i^\dagger = \Big(\prod_{j<i} \sigma_j^z\Big) \sigma_i^- \ .$$

   a. Show that the σ operators defined above satisfy the correct commutation relations, by using the fermion anti-commutation relations given above.

   For the problem at hand, it is more convenient to work in a transformed basis $\sigma^z \to \sigma^x$, $\sigma^x \to -\sigma^z$, so we have

   $$\sigma_i^x = (1 - 2c_i^\dagger c_i) \qquad\qquad \sigma_i^z = -\Big(\prod_{j<i} 1 - 2c_j^\dagger c_j\Big)(c_i^\dagger + c_i)$$

b. We will assume that the chain contains L sites, and is closed, so that the sites $i = L$ and $i = 0$ are identified. Show that after the Jordan-Wigner transformation, the hamiltonian reads

$$H_I = \sum_{i=0}^{L-1} g(2c_i^\dagger c_i - 1) + \sum_{i=0}^{L-2} \big(c_i c_{i+1} + c_i c_{i+1}^\dagger + c_{i+1} c_i^\dagger + c_{i+1}^\dagger c_i^\dagger\big) \\ - (-1)^F \Big(c_{L-1} c_0 + c_{L-1} c_0^\dagger + c_0 c_{L-1}^\dagger + c_0^\dagger c_{L-1}^\dagger\Big) ,$$

where F is the number of fermions, which is conserved modulo two.

The boundary term can be taken into account by imposing periodic boundary conditions $c_L = c_0$ when F is odd, and anti-periodic boundary conditions $c_L = -c_0$ when F is even. With these boundary conditions, the hamiltonian is translationally invariant, namely

$$H_I = \sum_{i=0}^{L-1} g(2c_i^\dagger c_i - 1) + \sum_{i=0}^{L-1} \big(c_i c_{i+1} + c_i c_{i+1}^\dagger + c_{i+1} c_i^\dagger + c_{i+1}^\dagger c_i^\dagger\big) .$$

c. To diagonalize the hamiltonian, we first have to go to momentum space, via $c_j = \frac{1}{\sqrt{L}} \sum_k c_k e^{iakj}$, with $a = \frac{2\pi}{L}$. Show that the hamiltonian takes the following form:

$$H_I = \sum_k 2(g - \cos(ak)) c_k^\dagger c_k + i \sin(ak)(c_{-k} c_k + c_{-k}^\dagger c_k^\dagger) - g .$$

To bring the hamiltonian in diagonal form, one performs a Bogoliubov transformation to a different set of fermions:

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger \qquad\qquad \gamma_k^\dagger = u_k c_k^\dagger + i v_k c_{-k} ,$$

where u_k and v_k are real, and satisfy $u_k^2 + v_k^2 = 1$, $u_{-k} = u_k$ and $v_{-k} = -v_k$, ensuring that the γ_k satisfy fermion anti-commutation relations.

d. Choose the parametrization $u_k = \cos\theta_k/2$ and $v_k = \sin\theta_k/2$, and find the condition on $\tan\theta_k$, such that terms which do not conserve the number of γ fermions (like $\gamma\gamma$) are absent in the hamiltonian, expressed in terms of the γ's.

e. Finally, show that the hamiltonian takes the form

$$H_I = \sum_k \varepsilon_k \big(\gamma_k^\dagger \gamma_k - \frac{1}{2}\big) \qquad\qquad \varepsilon_k = 2(1 + g^2 - 2g\cos(ak))^{\frac{1}{2}}$$

Note that for F odd, the momenta take the values $k = 0, 1, \ldots, L-1$, while or F even, the allowed momenta are $k = 1/2, 3/2, \ldots, L-1/2$, because of the boundary conditions.

2. The infinitesimal form of the special conformal transformation.

   Write the most general from of the quadratic part of ε_μ, namely $\varepsilon_\mu = c_{\mu\nu\rho} x^\nu x^\rho$, to derive the infinitesimal transformation

$$x'^\mu = x^\mu + b^\mu x^2 - 2x^\mu (b \cdot x) \ ,$$

   and give the explicit form of b^μ.

3. The (finite) special conformal transformations (in d dimensions) have the form

$$\mathbf{x}' = \frac{\mathbf{x} + \mathbf{b} x^2}{1 + 2\mathbf{b} \cdot \mathbf{x} + b^2 x^2} \ .$$

a. Derive the scale factor $\Omega(\mathbf{x})$ for this transformation: $\Omega(\mathbf{x}) = (1 + 2\mathbf{b} \cdot \mathbf{x} + b^2 x^2)^2$

b. Show that $|\mathbf{x}_1' - \mathbf{x}_2'|^2 = \frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{\gamma_1 \gamma_2}$, where $\gamma_i = (1 + 2\mathbf{b} \cdot \mathbf{x}_i + b^2 x_i^2)$.

4. 'Quasi-primary' fields transform under (passive) global conformal transformations in d dimensions as follows

$$\phi_i(\mathbf{x}) \to \left| \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \right|^{\frac{h_i}{d}} \phi_i(\mathbf{x}') \ ,$$

   where the h_i are the scaling dimensions of the fields ϕ_i.

   Show that the three point functions of quasi-primary fields have the following form

$$\langle \phi_1(\mathbf{x}_1) \phi_2(\mathbf{x}_2) \phi_3(\mathbf{x}_3) \rangle = \frac{C_{123}}{x_{12}^{h_1+h_2-h_3} x_{23}^{h_2+h_3-h_1} x_{13}^{h_1+h_3-h_2}} \ ,$$

   where we introduced the notation $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$.

5. (Optional). On the course website, http://www.nordita.org/˜ardonne/cft-course.html, there are three pictures. What do they correspond to?