Quantum Field Theory for Condensed Matter - 2018
Exercise Set 1 (14 points)
Due date: monday, april 16th

1. (0.5 p) Show that for a set of operators A, B and C

   a)
$$[AB, C] = A[B, C] + [A, C]B. \tag{1}$$
   This can be useful for bosonic systems.

   b)
$$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{2}$$
   This can be useful for fermionic systems.

2. (0.5 p) Fermion creation ($c^\dagger_\sigma(\mathbf{r})$) and annihilation ($c_\sigma(\mathbf{r})$) operators in real space obey the anti-commutation algebra,
$$\{c_\sigma(\mathbf{r}), c^\dagger_{\sigma'}(\mathbf{r}')\} = \delta_{\sigma\sigma'}\delta(\mathbf{r} - \mathbf{r}'), \qquad \{c_\sigma(\mathbf{r}), c_{\sigma'}(\mathbf{r}')\} = 0. \tag{3}$$
   Assume that the operator $a^\dagger_{i\sigma}$ creates an electron in the state $|W_i\rangle$ which is a Wannier state localized at site i of the lattice. Using a change of basis and the orthonormality of the Wannier states, show that these operators obey the fermionic algebra as well,
$$\{a_{i\sigma}, a^\dagger_{j\sigma'}\} = \delta_{\sigma\sigma'}\delta_{ij}, \qquad \{a_{i\sigma}, a_{j\sigma'}\} = 0. \tag{4}$$

3. a) (0.5 p) Consider a general quadratic Hamiltonian
$$H = \sum_{ij} a^\dagger_i \mathcal{H}_{ij} a_j, \tag{5}$$
   where $a^\dagger_i$ could be a boson or fermion creation operator. What is the constraint on $\mathcal{H}$ such that the hamiltonian H is hermitian?

   b) (0.5 p) Assume that a_i is either a boson or fermion annihilation operator. If we perform a linear transformation like,
$$d_i = \sum_j M_{ij} a_j, \tag{6}$$
   what requirement do we have to put on M in order to preserve the algebra, i.e. the operators d_i obey the same bosonic or fermionic algebra.
   Furthermore, show that the discrete Fourier transform,
$$a_{k\sigma} = \frac{1}{\sqrt{\mathcal{N}}} \sum_i e^{-i\mathbf{k}.\mathbf{r}_i} a_{i\sigma}, \tag{7}$$
   where $\mathcal{N}$ is the number sites and $\mathbf{r}_i$ is the position of the ith lattice point, meets the requirement.

4. a) (1.5 p) Take a square lattice of size $N \times N$ with lattice spacing a and set $\mathcal{H}_{ij} = -t$ for nearest neighbours and zero otherwise in the Exercise 3.a) for a fermionic system. Using a discrete Fourier transformation calculate eigenvalues of the Hamiltonian $\epsilon_{\mathbf{k}}$.

   b) (0.5 p) Sketch the contours of constant $\epsilon_{\mathbf{k}}$ in the Brillouin zone and note the geometry at half-filling (i.e., the Fermi surface obtained by filling exactly half of all available states).

5. Do exercise 3.3 from Coleman (2.0 p).

6. Jordan-Wigner transformation.

a) (1.0 p) Starting from the Jordan-Wigner transformation Eq. (4.9) in Coleman, express the fermion creation and annihilation operators in terms of Pauli operators (including the raising and lowering operators), and show that they satisfy the canonical anti-commutation relations, by using the properties of the Pauli matrices.

b) (1.0 p) We consider the following *periodic* hamiltonian with L sites (i.e., $\sigma^\alpha_{j+L} = \sigma^\alpha_j$):

$$H = \sum_{j=1}^{L} J_x \sigma^x_j \sigma^x_{j+1} + J_y \sigma^y_j \sigma^y_{j+1} + h_z \sigma^z_j \ . \tag{8}$$

Rewrite this hamiltonian in terms of fermionic degrees of freedom. Pay special attention to the 'boundary term', and discuss its form.

7. Consider a general linear transformation, a so called Bogoliubov transformation,

$$d_i = \sum_j A_{ij} c_j + B_{ij} c^\dagger_j.$$

a) (1.0 p) Show that the requirements for d_i to have the same bosonic or fermionic algebra as c_i, *i.e.*,

$$\left[d_i, d^\dagger_j\right]_\pm = \delta_{ij}, \qquad [d_i, d_j]_\pm = 0,$$

where $[\,,\,]_-$ denotes a commutator and $[\,,\,]_+$ denotes an anti-commutator, are the following ones:

$$AA^\dagger \pm BB^\dagger = 1 \qquad AB^T \pm BA^T = 0,$$

where $+$ is fermions and $-$ for bosons.

b) (1.0 p) For the fermionic case, show that AB^T is antisymmetric and that the first requirement in part a) can be satisfied by setting

$$A_{ij} = \cos\theta_i U_{ij}, \qquad B_{ij} = \sin\theta_i V_{ij}$$

where U and V are unitary matrices and the θ_i are real.

c) (1.0 p) For the bosonic case, show that AB^T is symmetric and that the the first requirement in part a) can be satisfied by setting

$$A_{ij} = \cosh\theta_i U_{ij}, \qquad B_{ij} = \sinh\theta_i V_{ij}$$

where U and V are unitary matrices and the θ_i are real.

8. Do exercise 2.7 from Coleman (3.0 p).