Ginzburg-Landau for charged fields:
super conductors.

Neutral superfluid: twist in phase → superflow
charged " : phase twist → super current.

Super conductors expel the magnetic field:
Meissner effekt.

Gauge field changes the Hamiltonian:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \left( \vec{\nabla} - i \frac{e}{\hbar} \vec{A} \right)^2 + e\,\varphi(x) \right] \psi$$

OBS: $e = -1.6 \cdots 10^{-19} C$

Gauge invariance: $\psi(x,t) \to e^{i\alpha(x,t)} \psi(x,t)$

$$\vec{A} \to \vec{A} + \frac{\hbar}{e} \nabla \alpha \;;\; \varphi \to \varphi - \frac{\hbar}{e} \frac{\partial \alpha}{\partial t}$$

Same in GL (etc!), but we change $\vec{\nabla} \to \vec{D} = \vec{\nabla} - \frac{i e^*}{\hbar} \vec{A}$,
where $e^*$ is charge of condensing field
(SC: cooper pair, $e^* = 2e$).

GL free energy for SC: in terms of $\vec{D}$ and add energy
for magnetic field!

$$F[\psi, A] = \int d^d x \left[ \frac{\hbar^2}{\underbrace{2M}_{2m_e}} \left| \left( \vec{\nabla} - \frac{i e^*}{\hbar} \vec{A} \right) \psi \right|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 + \underbrace{\frac{(\vec{\nabla} \times \vec{A})^2}{2\mu_0}}_{f_{EM}} \right]$$

~~Mse~~
lm

invariant under $\psi(x) \to \psi(x) e^{i\alpha(x)}$

$$A \to A + \frac{\hbar}{e^*} \nabla\alpha$$

We get "two free energies" for $\psi$ and $A$:

Coherence length: $\xi = \sqrt{\frac{\hbar^2}{2M|\alpha|}}$

Penetration depth: for $\psi = \sqrt{n_s}$, we have

$$f[A] = c_A \left(\frac{\vec{\nabla}\times\vec{A}}{2}\right) + \frac{r_A}{2} A^2,$$

$$c_A = \frac{1}{\mu_0}, \quad r_A = \frac{(e^*)^2 n_s}{M}, \text{ so } \lambda_L = \sqrt{\frac{c_A}{r_A}} = \sqrt{\frac{M}{n_s (e^*)^2 \mu_0}}$$

Varying the GL free energy wrt $A$ and $\psi$ gives GL equations: give rise to Meissner effect and ~~to~~ domain walls in type II SCs!

$$F = \cancel{F[\psi]}\ F_\psi + F_{EM}, \text{ vary } \vec{A} \text{ gives}$$

$$\delta F_\psi = -\int d^dx\, \delta\vec{A}(x) \cdot \underbrace{\left[ -\frac{ie^*\hbar}{2M}\left(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) - \frac{(e^*)^2}{M}|\psi|^2 \vec{A} \right]}_{\vec{j}(x)}$$

$$\delta F_{EM} = \frac{1}{\mu_0}\int d^dx\, \delta\vec{A}(x) \cdot (\vec{\nabla}\times\vec{B})$$

careful!

Use components!

So we obtain: $\frac{\delta F}{\delta \vec{A}} = -\vec{J}(x) + \frac{\vec{\nabla}\times\vec{B}}{\mu_0} = 0$

$$\vec{j}(x) = -\frac{i e^* \hbar}{2M}\left(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) - \frac{(e^*)^2}{M}|\psi|^2\vec{A}$$

supercurrent density

~~This~~ is Ampere's law, ~~pr~~ but $\psi(x)$ macroscopic wave function of superconducting condensate.

↳ Gives Meissner effect!

Vary wrt $\psi(x)$:

$$\delta F = \int d^d x \left\{ \delta\psi^*(x)\left[\frac{\hbar^2}{2M}\left(-i\vec{\nabla} - \frac{e^*}{\hbar}\vec{A}\right)^2\psi(x) + r\,\psi(x) + u|\psi|^2\psi(x)\right] + h.c.\right\}$$

or: $-\frac{\hbar^2}{2M}\left(\vec{\nabla} - i\frac{e^*}{\hbar}\vec{A}\right)^2\psi(x) + r\,\psi(x) + u|\psi(x)|^2\psi(x) = 0$

same as in Ising case, with: $\nabla^2 \to \left(\nabla - i\frac{q}{\hbar}\vec{A}\right)^2$ (Nonlin SE!)

Gives rise to surface tension of a drop of SC.

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Meissner effect: write ~~SC Amp~~ supercurrent $j(x)$ in term of amplitude and phase: $\psi = |\psi|e^{i\phi}$, which gives:

$$\vec{j}(x) = \frac{e^*\hbar}{M}|\psi|^2\vec{\nabla}\phi - \frac{(e^*)^2}{M}|\psi|^2\vec{A}$$

$$= e^{*} \overset{|\psi|^2}{\overset{\shortparallel}{n_s}} \underbrace{\frac{\hbar}{M}\left(\vec{\nabla}\phi - \frac{e^{*}}{\hbar}\vec{A}\right)}_{\vec{v}_s} = e^{*} n_s \vec{v}_s$$

Both twisting phase and vector potential give a ~~curren~~ current! OBS: invariant under $\phi \to \phi + \alpha$; $\vec{A} \to \vec{A} + \frac{\hbar}{e^{*}} \nabla\alpha$

Ampere's eq: $\vec{\nabla} \times \vec{B} = -\mu_0 \frac{n_s (e^{*})^2}{M} \left(\vec{A} - \frac{\hbar}{e^{*}} \vec{\nabla}\phi\right)$

Taking $\vec{\nabla}\times$ gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\underset{0}{\underset{\shortparallel}{\vec{\nabla}\cdot\vec{B}}}) - \nabla^2 \vec{B} = -\vec{\nabla}^2 \vec{B}$$

$$= -\mu_0 \frac{n_s (e^{*})^2}{M} \left[ (\vec{\nabla} \times \vec{A}) - \frac{\hbar}{e^{*}} \underset{0}{\underset{\shortparallel}{\vec{\nabla} \times \vec{\nabla}\phi}} \right]$$

So, we get: $\boxed{\vec{\nabla}^2 \vec{B} = \frac{1}{\lambda_L^2} \vec{B}}$ $\quad \lambda_L^2 = \frac{M}{\mu_0 n_s (e^{*})^2}$

Homogeneous solutions:

$\vec{B} = 0,\ n_s > 0 \qquad \vec{B} \neq \vec{0},\ n_s = 0.$

One-dim solution of $\nabla^2 B = B/\lambda_L^2$ gives $B \sim B_0 e^{-x/\lambda_L}$

B ~~deg~~ decays exponentially into condensate!

Screening currents are at surface of the SC!

Type II SC: $\xi < \lambda_L / \sqrt{2}$ :

B-field can penetrate SC in non homogeneous way: vortices.

Critical field $H_c$:

$$\vec{B} = \mu_0 (\underbrace{\vec{H}}_{\text{coil}} + \underbrace{\vec{M}}_{\text{magnetisation}})$$

$j_{ex} = \vec{\nabla} \times \vec{H}$

$j_{int} = \vec{\nabla} \times \vec{M}$

$\chi = M/H$ susceptibility

$\vec{B} = \vec{0}$ in SC, so $\vec{M} = -\vec{H} \Rightarrow \chi_{sc} = -1$ !

Perfect diamagnet

There is upper limit (Why?):

Better to use Gibbs free energy in terms of $H$, instead of $\vec{B}$

$$G[H, \psi] = F[B, \psi] - \int d^d x \, \vec{B} \cdot \vec{H} \qquad \left( \frac{\delta G}{\delta B} = 0 \right)$$

Uniform case: $g = \dfrac{G}{V} = r|\psi|^2 + \dfrac{u}{2} |\psi|^4 + \dfrac{B^2}{2\mu_0} - BH$

Normal state: $\psi = 0$; $B = \mu_0 H$ so $g_n = -\dfrac{\mu_0}{2} H^2$

SC state $\quad |\psi| = \psi_0 = \sqrt{\dfrac{-r}{u}}$ ; $B = 0$

$$g_{sc} = r \psi_0^2 + \frac{u}{2} \psi_0^4 = -\frac{r^2}{2u}$$

$g_{sc} < g_n$ if $H < H_c = \sqrt{\dfrac{r^2}{\mu_0 u}}$, so

$$g_{sc} = -\frac{\mu_0}{2} H_c^2$$

$\dfrac{\delta F_\psi}{\delta B} = -M$

$\dfrac{\delta F_{EM}}{\delta B} = B/\mu_0$

$\Rightarrow \dfrac{\delta F}{\delta B} = \dfrac{B}{\mu_0} - M = H$

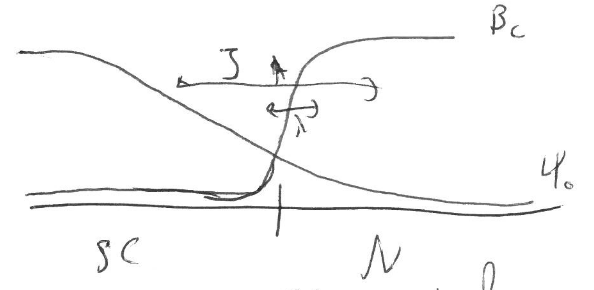
Energy of SC - normal interface:

For $H = H_c$, free energies of SC & normal metal are the same: coexistence.

Interface: domain wall w/ surface energy (tension)

$$\Delta g / A = \sigma_{sn}$$

$\uparrow$ interface




$\xi \gg \lambda$

$\sigma_{sn}$ can be positive or negative: different behaviours!

$K = \frac{\lambda L}{\xi}$ crucial parameter.

I; $K < \frac{1}{\sqrt{2}}$ $\sigma_{sn}$ positive, B expelled 'completely' (only outer surface)

II; $K > \frac{1}{\sqrt{2}}$ $\sigma_{sn} < 0$ : two critical fields:

upper $H_{c_2} > H_c$; lower $H_{c_1} < H_c$

For $H_{c_1} < H < H_{c_2}$: B penetrates via vortices (which form a lattice): High energy of normal core region comp. by neg. surface energy

$H < H_{c_1}$: no B-field penetrates.

Energy of domain-wall between SC and normal metal w/ $H = H_c$:

$$\sigma_{sn} = \frac{1}{A} \int d^3x \left[ \frac{\hbar^2}{2M} \left| (\vec{\nabla} - \frac{i\alpha}{\hbar} \vec{A}) \Psi \right|^2 + \alpha |\Psi|^2 + \frac{u}{2} |\Psi|^4 + \underbrace{\frac{B^2}{2\mu_0} - B H_c - g_{sc}} \right]$$

w/ $H_c = \frac{B_c}{\mu_0}$, $g_{sc} = -\frac{B_c^2}{2\mu_0}$, : $\frac{(B - B_c)^2}{2\mu_0}$

Look at example the interface: $x=0$ ($y$-$z$ plane):

$x<0$ SC; $x>0$ normal; $B$ in $\hat{z}$ direction.

~~Varying~~ Varying $\sigma_{SN}$ wrt $\psi$ and $A$ gives two eqs (one is London eq).

Using these, we can write (exercise! example):

$$\sigma_{SN} = \frac{B_c^2}{2\mu_0}\int_{-\infty}^{\infty} dx \left[ \underbrace{\left(\frac{B(x)}{B_c} - 1\right)^2}_{\text{field energy}} - \underbrace{\left(\frac{\psi(x)}{\psi_0}\right)^4}_{\text{condensation energy}} \right]$$

If condensation energy is larger than field energy (short healing length $\xi$) $\sigma_{SN} < 0$, i.e. type II, we get vortices (long $\lambda_L$).

Opposite for type I.

$\sigma_{SN}$ is zero when $K = \frac{1}{\sqrt{2}}$, see example in book.

Vortices appear in type II SC, for $H > H_{c_1}$: defect lines at which $\psi = 0$.

Phase of order parameter winds around them, as in SF!

Supercurrents gen. B field, $\Phi_0 = \frac{h}{e^*} = \frac{h}{2e}$: quantised flux!

Super fluid: $\vec{V}_s = \frac{\hbar}{M} \vec{\nabla}\phi$

Superconductor: $\vec{V}_s = \frac{\hbar}{M} \vec{\nabla}\phi - \frac{e^*}{M}\vec{A}$

Vector pot. contribution reduces contribution from $\vec{\nabla}\phi$.

The circulation: $\omega = \oint dx \cdot V_s = \frac{\hbar}{M} \underbrace{\oint d\vec{x} \cdot \vec{\nabla}\phi}_{\Delta\phi = 2\pi n} - \frac{e^*}{M} \underbrace{\oint d\vec{x} \cdot \vec{A}}_{\Phi \leftarrow \text{flux}}$

So we get $\omega = n\frac{h}{M} - \frac{e^*\Phi}{M}$

Far away from vortex $\omega \to 0$ (from energetics) so

$$\Phi = n\left(\frac{h}{e^*}\right) = n\Phi_0$$ : flux is quantised!

Vortex w/ lowest energy: one flux quantum.

Take hollow cylinder as the SC: lowest energy state, no supercurrent, integer # flux quanta.

Cool SC below transition in field, remove field and measure the flux: quantised in units of $\phi_0 = \frac{h}{2e}$

demonstration of flux quantisation, and presence of cooper pairs!!