Variations in the orderparameter:

Ginzburg-Landau theory.

One component case:

$$f_{GL}[\psi, \nabla\psi] = \frac{S}{2}|\nabla\psi|^2 + f_L[\psi]$$

$$F_{GL} = \int d^dx\, f_{GL}[\psi(x), \nabla\psi(x), h(x)]$$

$$\mathcal{L} = \frac{S}{2}(\nabla\psi)^2 + \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 - h\psi$$

Valid near critical point!

Dimensions: $[S/r] = L^2$, so $\xi(T) = \sqrt{\frac{S}{|r(T)|}}$ is a length, correlation length

$$\xi(T) = \xi_0 \left|1 - T/T_c\right|^{-\frac{1}{2}}, \quad \xi_0 = \xi(T=0) = \sqrt{\frac{S}{aT_c}}$$

= coherence length.

$\uparrow$

diverges at T_c

One uses GL theory as a variational theory so using $\frac{\delta F}{\delta \psi} = 0$.

Non-uniform solutions:

* linear, non-local respons to small external fields
* soliton solutions: ψ changes sign at domainwall.

Variation:

$$\delta F_{GL} = \int d^d x \; \delta\psi(x) \left[-s \nabla^2 \psi(x) + \frac{\partial f_L[\psi]}{\partial \psi(x)} \right]$$

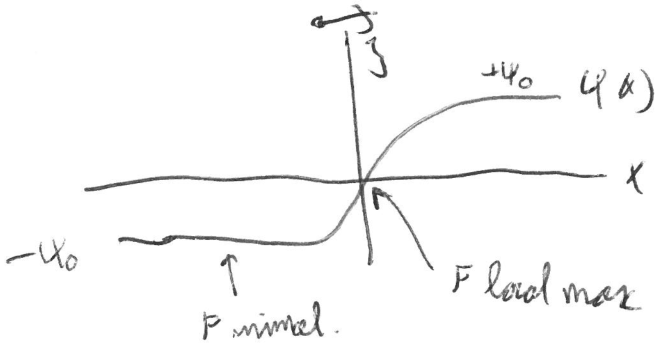
F_{GL} stationary under small perturbations:

$$\Rightarrow \quad -s \nabla^2 \psi(x) + \frac{\partial f_L[\psi]}{\partial \psi(x)} = 0, \text{ or}$$

$$\left[\left(-s \nabla^2 + r\right) + u\,\psi^2(x) \right] \psi(x) - h(x) = 0$$

Domain walls: * deviations from ψ_0 are costly if $T < T_c$. Can occur at domain walls (solitons)

↳ separat two regions w/ $\psi = \pm\psi_0$



In one dimension: $s\,\psi'' = \dfrac{d f_L[\psi]}{d\psi}$ (*)

particle w/ mass s, in potential $V[\psi] = -f_L[\psi]$

ψ is like a displacement, $S/2(\psi')^2$ is 'kinetic energy':

$$\mathcal{E} = \frac{S}{2}(\psi')^2 - f_L[\psi]$$

Multiply $(*)$ by ψ':

$$S\psi''\psi' - \psi'\frac{df}{d\psi} = \frac{d}{dx}\left[\frac{S}{2}(\psi')^2 - f_L[\psi]\right] = 0.$$

The initial conserved energy is $\mathcal{E} = -f_L[\psi_0] = \frac{r^2}{4u}$ $(r<0)$!

$$\psi' = \frac{d\psi}{dx} = \sqrt{\frac{2}{S}\left(\mathcal{E} + f_L[\psi]\right)} = \frac{\psi_0}{\sqrt{2}\xi}\left(1 - \frac{\psi^2}{\psi_0^2}\right)$$

Integrating gives:

$$(x - x_0) = \frac{\sqrt{2}\xi}{\cancel{\psi_0}}\int_0^{\psi} \frac{d(\psi/\psi_0)}{1-(\psi/\psi_0)^2} = \sqrt{2}\xi\tanh^{-1}\left(\frac{\psi}{\psi_0}\right)$$

So: $\psi(x) = \psi_0 \tanh\left(\frac{x - x_0}{\sqrt{2}\xi}\right)$ Soliton solution!

Complex order & superflow in GL

GL for superfluids & superconductors:
complex order parameters.

Bold step: expectation value of $\hat{\psi}(x)$ is
a 'macroscopic wavefunction':

$$\langle \hat{\psi}(x) \rangle = |\psi(x)| e^{i\phi(x)}, \text{ where}$$

$|\psi(x)|^2 = n_s(x)$: superfluid density

$\vec{v}_s(x) = \frac{\hbar}{m} \vec{\nabla}\phi(x)$: s.f. velocity

Copenhagen: wavefunction not observable.

Here: $\psi(x)$: collective, emergent macroscopic object

GL free energy for superfluid:

$$f_{GL}[\psi, \nabla\psi] = \frac{\hbar^2}{2m} |\vec{\nabla}\psi|^2 + r|\psi|^2 + \frac{u}{2} |\psi|^4$$

↳ free energy density of condensate of bosons, $\langle \hat{\psi}(x) \rangle$: complex order parameter.

$s|\vec{\nabla}\psi|^2 = \frac{\hbar^2}{2m} \langle \nabla\psi^{+} \nabla\psi \rangle$; kinetic energy

~~Coherence~~ Correlation length: $\xi = \sqrt{\frac{S}{|r|}} = \xi_0 \left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}}$

$$\downarrow$$

$$= \sqrt{\frac{\hbar^2}{2maT_c}}$$

If we neglect amplitude fluctuations:

$$\psi(x) = \sqrt{n_s}\, e^{i\phi(x)} \rightarrow \nabla\psi = i\nabla\phi\,\psi,$$

$$|\nabla\psi|^2 = n_s(\nabla\phi)^2$$

Kin energy $\frac{\hbar^2}{2m} n_s (\nabla\phi)^2 = \frac{mn_s}{2}\underbrace{\left(\frac{\hbar}{m}\nabla\phi\right)^2}_{v_s^2}$

v_s ↳ superfluid velocity.

What is the meaning of ψ?

Well: $\langle\hat{\psi}(x)\rangle = \psi$, but $\langle\hat{\psi}(x)\rangle = 0$ for system w/ definite particle number.

Two alternatives:

* ODLRO: $\langle\psi^\dagger(x')\psi(x)\rangle \xrightarrow{|x'-x| \gg \xi} \psi^*(x')\psi(x) + \ldots$, signals broken symmetry in ~~pt~~ state w/ definite particle number.

* Consider states w/ particle number fluctuations. (~~be~~ system in contact w/ bath of particles, cooling of magnet w/ field in x direction $-\sigma^z\sigma^z$

Description of ~~[illegible]~~ such states:

Coherent states: eigenstate of $\hat{\psi}(x)$:

$$\hat{\psi}(x)\,|\psi\rangle = \psi(x)\,|\psi\rangle$$

(Why not of $\hat{\psi}^\dagger(x)$?)

$$|\psi\rangle \propto e^{\sqrt{N_s}\, b^\dagger}\,|0\rangle \qquad b^\dagger = \frac{1}{\sqrt{N_s}}\int d^dx\, \psi(x)\,\hat{\psi}(x)$$

adds boson to condensate w/ w.f. $\psi(x)$.

Coherent state: many ~~do~~body eq. of wave packets.

$(\hat{p}, \hat{x}) \rightarrow (N, \phi)$:

$$e^{i\alpha\hat{N}}\,|\phi\rangle = |\phi+\alpha\rangle, \text{ or } \langle\phi+\delta\phi| = \langle\phi|\,1 - i\delta\phi\hat{N},$$

$$\text{so } i\frac{d}{d\phi}\langle\phi| = \langle\phi|\,\hat{N},$$

$$\hat{N} = i\frac{d}{d\phi},$$

Which means $[\hat{N}, \hat{\phi}] = i$, or $\Delta\phi\,\Delta N \gtrsim 1$

Which implies that ~~[illegible]~~ for $N \sim 10^{23}$, we can know N and ϕ, to order 10^{-11}

So, very accurate description of energy eigenstates at high density

Expectation value of normal ordered operators in coherent states: ~~replace field~~ replace operators by order parameter!

Phase rigidity:

Effect of twisting the phase:

$$\psi = |\psi| e^{i\phi} \Rightarrow \nabla\psi = (\nabla\psi + i\nabla\phi|\psi|)e^{i\phi}, \text{ so}$$

$$f_{GL} = \underbrace{\frac{\hbar^2}{2m}|\psi|^2(\nabla\phi)^2}_{\text{phase rigidity}} + \underbrace{\frac{\hbar}{2m}(\nabla\psi)^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4}_{\text{amplitude fluct.}}$$

amplitude fluctuations occur only up the length of order correlation length. Longer scale: phase degree of freedom:

$$f_{GL} = \frac{\rho_\phi}{2}(\nabla\phi)^2 + \text{const}$$

$$\rho_\phi = \frac{\hbar^2}{2m} n_s = \text{phase stiffness}$$

Microscopic: kin energy of particles

Macro: elastic term $\Rightarrow$ twist of phase gives coherent flow of particles $\vec{J}$.

current operator $\vec{J} = -i\frac{\hbar}{2m}(\hat{\psi}^+\nabla\hat{\psi} - (\nabla\hat{\psi}^+)\hat{\psi})$, or

$$\langle\psi|J|\psi\rangle = -\frac{i\hbar}{2m}(\psi^*(\nabla\psi) - (\nabla\psi^*)\psi), \text{ using}$$

$$\psi(x) = \sqrt{n_s}\, e^{i\phi(x)}$$

gives

$$\vec{J}_s = \frac{n_s\hbar}{m}\vec{\nabla}\phi = n_s\vec{v}_s \quad \vec{v}_s = \frac{\hbar}{m}\vec{\nabla}\phi$$

Twisting phase sets all particles in motion...

Vorticity:

Stability of superflow: topology of phases! 

On cylinder: phase well defined: $\Delta\phi = \oint d\vec{x}\cdot\vec{\nabla}\phi$

$$= 2\pi n_\phi$$

But: $v_s = \frac{\hbar}{m}\vec{\nabla}\phi$, so $\omega = \oint d\vec{x}\cdot\vec{v}_s = \frac{h}{m} n_\phi$,

quantisation of 'circulation' ω.

W/ translation invariance: $v_s = \frac{h}{mL} n_\phi$,

quantisation of velocity!

Note: n_ϕ: topological invariant, can only be changed by high-energy domain walls.

Vortex: singular line in superfluid, phase winds around it

path w/ radius r around vortex:

$$\omega = n_\phi\left(\frac{h}{m}\right) = \oint dx\, v_s(x) = 2\pi r v_s \text{, so}$$

$$v_s = n_\phi\left(\frac{\hbar}{m}\right)\frac{1}{r}, \quad \text{if } r \gg \xi$$

$$\nabla\phi = \frac{v_s m}{\hbar} = \frac{n_\phi}{r}$$

Free energy of vortex in cylinder of radius R:

$$\frac{F}{L} = \frac{\rho_\phi}{2}\int d^2x\,(\nabla\phi)^2 = \frac{\rho_\phi}{2}\int_\xi^R \cancel{2\pi r\,dr}\, dr\,(2\pi r)\left(\frac{n_\phi}{r}\right)^2$$

$$= \rho_\phi \pi n_\phi^2 \ln(R/\xi).$$

Vortex of high winding $\pm n_\phi$ ~~to~~ splits in n_ϕ vortices w/ winding ± 1!