Phase transitions, order parameters,

broken symmetry etc. (not: top. phases)

Broken symmetry is all around us:

* magnets
* solids
* superconductors / BEC
* liquid crystals
* current space-time

⋮

Can be described in terms of unifying concept, despite different microscopics: order parameter

$$|\psi| = \begin{cases} 0 & T > T_c \\ |\psi_0| > 0 & T < T_c \end{cases}$$

ψ: scalar, vector, spinor

ψ is related to quantum operator: Ising magnet $m = \langle \sigma_z(x) \rangle$

Superfluid: exp. value of bos. field $(\in \mathbb{C})$

~~Exp~~ When $|\psi| > 0$, symmetry is broken

(snow flakes: cont. sym $\to$ 6-fold sym)

(Ferromagnet: magnetisation, SC: expells mag fields etc.)

Landau theory: valid down to length scales of order ξ_0: coherence length.
Smaller length scales: need microscopic theory.

Landau theory:

Idea: write free energy as $F[\psi]$
Start simple: scalar order parameter (Ising).

We can induce the ~~OP~~ O.P. by cooling in a field h that couples to the O.P.
↳ Broken symmetry: ψ remains finite if we set h to zero.

Introduce the field: $H \to H - h \int d^3x\, \hat{\psi}(x)$

$\uparrow$ conjugate field

Ex: $h = B$ if $\psi = M$ (magnetisation)
$h = E$ if $\psi = P$ (polarisation)

h is not always an 'easily available' field, [anti-ferromagnets, SC, etc].

Take ~~field~~ field into account via Gibbs free energy:

$$G[h] = F[\psi] - V\psi h$$

$$\mathcal{G}[h] = -k_B T \ln [Z[h]]$$
$$= -k_B T \,\mathrm{Tr}\left(e^{-\beta[H - h\int d^3x\, \hat{\psi}(x)]}\right)$$

Expectation value of order parameter:

$\psi[h] = \langle \hat{\psi} \rangle$: diff with h

$$\psi(h, V) = \frac{1}{Z[h]} \mathrm{Tr}\left[e^{-\beta[H - h\int d^3x\, \hat{\psi}]}\, \hat{\psi}(x)\right] = -\frac{1}{V}\frac{\partial \mathcal{G}}{\partial h}$$

Finite systems: O.P. will disappear if h is
(microscopic) set to zero, because of thermal fluctuations

[~~not~~ small cluster of spins: cool below T_c in field: remove field. T will induce domain ~~walls~~ w reversed spins.
For large times, mag. will go to zero.
Now: take macroscopic system: * infinitesimal fields prevent macroscopic domains
* in $h=0$, probability for domains to form becomes 'zero']

So: stable phase: take $V \to \infty$ limit before sending $h \to 0$.

Thus: $\psi = \lim_{h \to 0} \lim_{V \to \infty} \psi(h, V)$

Landau free energy:

Around phase-transition, ψ is small, so let's expand:

$$f_L[\psi] = \frac{1}{V} F[\psi] = \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4$$

Total F: contains ψ independent part:

$$f_{tot} = \underset{\substack{\uparrow \\ \text{normal state}}}{f_n(T)} + f_L[\psi] + O(\psi^6)$$

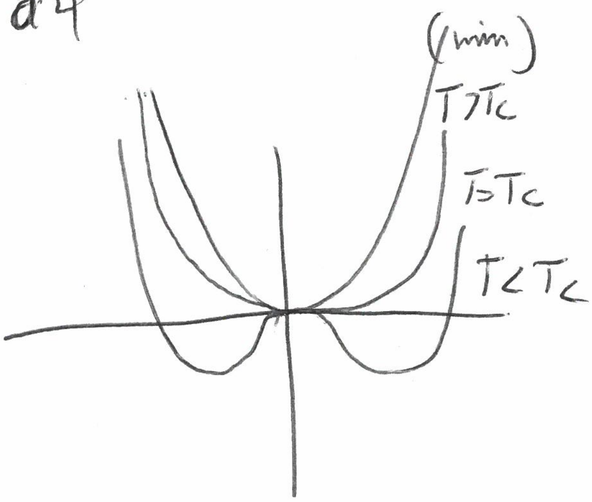
Ising case: $H[\psi] = + H[-\psi]$, $f_L[-\psi] = f[\psi]$:

Z_2 symmetry (invariant under $\psi \to -\psi$)

For $r, u > 0$: minimum of f is at $\psi = 0$.

Landau: r changes sign at T_c: $r = a(T - T_c)$

$$\frac{\partial f_L}{\partial \psi} = r\psi + u\psi^3 = 0 \Rightarrow \psi_0 = \begin{cases} 0 & T > T_c \\ \underset{\substack{\uparrow \\ \text{sign } h}}{\pm}\sqrt{\dfrac{a(T_c - T)}{u}} & T < T_c \end{cases}$$



Sign of ψ_0: set by sign of h if cooled in a field h.

So: branch-cut along T ~~axes~~ axis ($h=0$).
(First order transition)

Branch cut ends at $T=T_c$, $h=0$: critical point.

* if $u<0$: need to add sixth order term to $f_L[\varphi]$

$$f_L[\varphi] = \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 + \frac{u_6}{6}\varphi^6$$

Three minima if $u<0$, First order transition.

$r=h=u=0$: tri-critical point.

Singularities at critical point:

Free energy in term of T:

$$f_L = \begin{cases} 0 & T > T_c \\ -\frac{a^2}{4u}(T_c-T)^2 & T < T_c \end{cases}$$

So, both f_L and $S = -\frac{\partial f}{\partial T}$ are continuous at T_c,

but $C_V = -T\frac{\partial^2 f_t}{\partial T^2} = C_0(T) + \begin{cases} 0 & T>T_c \\ \frac{a^2 T}{2u} \end{cases}$

is discontinuous: $\Delta C_V = \frac{a^2 T}{2u}$; entropy per volume

h plays an important role at T_c: large susceptibility!

$\chi = \frac{\partial \psi}{\partial h}$ diverges

$$f_2(\psi) \to f_2(\psi, h) = \frac{r}{2}\psi^2 + \frac{u}{4}\psi^4 - h\psi$$

$h > 0$: 'right' minimum is deeper.

$$0 = \frac{\partial f}{\partial \psi} = r\psi + u\psi^3 - h = 0$$

To find $\chi(T)$, we linearise $\psi[h] = \psi_0 + \delta\psi$

This gives: $\delta\psi = \chi(T)\, h$ + h.o.t., where

$$\chi(T) = \frac{\partial \psi}{\partial h} = \frac{1}{a|T-T_c|} \cdot \begin{cases} 1 & T > T_c \\ \frac{1}{2} & T < T_c \quad (h < 0?) \end{cases}$$

χ diverges at T_c!

~~Same for~~ ψ itself: $\psi = \left(\frac{h}{u}\right)^{\frac{1}{3}}$ $(T = T_c)$ is non-linear!

When cooling, stray field will align the spins: we ~~say~~ say that system spontaneously breaks the $\mathbb{Z}_2$ symmetry.

Phase transition (critical point): described by scaling exponents:

$C_V \propto |T-T_c|^{-\alpha}$

$\psi \propto \begin{cases}(T_c-T)^{\beta} \\ h^{1/\delta}\end{cases}$

$\chi \propto (T-T_c)^{-\gamma}$

In Landau theory: $\alpha = 0;\ \beta = \frac{1}{2};\ \gamma = 1;\ \delta = 3$.

Generic phase transitions: can have different exponents, but they fall in 'universality classes', independent of microscopic details!

Continuous symmetries:

n-comp. order parameter $\vec{\psi} = (\psi_1, \dots, \psi_n)$ $O(n)$ rot symm!

Replace ψ^2 by $\vec{\psi}\cdot\vec{\psi} = |\psi|^2$

$f_L[\psi] = \frac{r}{2}(\vec{\psi}\cdot\vec{\psi}) + \frac{u}{4}[(\vec{\psi}\cdot\vec{\psi})]^2$, w/ $r = a(T-T_c)$

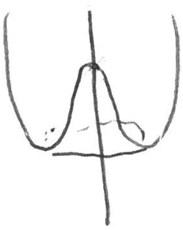
↑ inv. under $\vec{\psi} \to R\vec{\psi}$

For $T < T_c$: $\vec{\psi} = \sqrt{\frac{|r|}{u}}\,\hat{n}$: spont. mag. in a magnet.

Complex order parameter $\psi = \psi_1 + i\psi_2 = |\psi| e^{i\phi}$

$f_L[\psi] = r(\psi^*\psi) + \frac{u}{2}(\psi^*\psi)^2$ (factor 2: convention)

↳ $\psi \to e^{i\alpha}\psi$; invariance!



mexican hat! $\psi_0 = \sqrt{\frac{|r|}{u}}\, e^{i\phi}$

If well def. phase appears: $U(1)$ sym is broken!

Need phase rigidity for SC, ~~supe~~ superfluids.

If we twist the phase, supercurrents appear: $\vec{J} \propto \vec{\nabla}\phi$