Finite Temperature

many body physics

At finite T: all states ~~trace~~ contribute to physical properties:

$$p_\lambda = \frac{e^{-\beta E_\lambda}}{Z}$$

Remarkably: <u>not</u> more complicated than $T=0$!

QM part. function: $\hat{\rho} = e^{-\beta \hat{H}} = U(-i\hbar\beta)$,

where $U(t) = e^{-iHt/\hbar}$ is time evolution operator;

$H = H_0 - \mu N$ ~~(Kubo)~~

Kubo: finite T: use imag time: $\frac{it}{\hbar} \to \tau$

Matsubara: $Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, U(-i\beta\hbar)$

↳ U evaluated at $t = -i\hbar\beta$

Expectation values:

$$\langle A \rangle = \frac{\mathrm{Tr}[U(-i\hbar\beta) A]}{\mathrm{Tr}[U(-i\hbar\beta)]}$$, like Gell-Mann–Low,

but S-matrix replaced by time evolution over a finite interval $t \in [0, -i\hbar\beta]$, or $\tau \in [0, \beta\hbar]$

Physics will be periodic ~~in~~ w/ this time interval

Time scale of thermal fluctuations:

$$\tau_T = \frac{\hbar}{k_B T}$$ On longer time scale: coherence lost.

Using $it \to \tau$, we can proceed as in $T=0$ case!

$$H|\psi_S\rangle = i\hbar \frac{\partial}{\partial t}|\psi_S\rangle \leftrightarrow H|\psi_S\rangle = -\frac{\partial}{\partial \tau}|\psi_S\rangle$$

Heisenberg: $|\psi_H\rangle = |\psi_S(0)\rangle$:

$$A_H(\tau) = e^{iH(-i\tau)} A_S e^{-iH(-i\tau)} = e^{H\tau} A_S e^{-H\tau}$$

giving EOM: $$\frac{\partial A_H}{\partial \tau} = [H, A_H]$$

For free particles: $H = \sum_k \varepsilon_k c_k^\dagger c_k$:

$$\frac{\partial c_k}{\partial \tau} = [H, c_k] = -\varepsilon_k c_k$$

$$\frac{\partial}{\partial \tau} c_k^\dagger = [H, c_k^\dagger] = \varepsilon_k c_k^\dagger$$

So $c_k(\tau) = e^{-\varepsilon_k \tau} c_k$

$c_k^\dagger(\tau) = e^{\varepsilon_k \tau} c_k^\dagger$

OBS: $c_k^\dagger(\tau) = [c_k(-\tau)]^\dagger \neq [c_k(\tau)]^\dagger$

Interaction picture: remove non-int part of evolution.

$$|\psi_I(\tau)\rangle = e^{H_0\tau}|\psi_S(\tau)\rangle = e^{H_0\tau}e^{-H\tau}|\psi_H\rangle$$

$$= \underbrace{U(\tau)}_{\text{'time' evolution operator}}|\psi_H\rangle$$

We obtain $A_H(\tau) = e^{H\tau}A_S e^{-H\tau} = U^{-1}(\tau)A_I(\tau)U(\tau)$

Evolution of states:

$$|\psi(\tau_1)\rangle = U(\tau_1)U^{-1}(\tau_2)|\psi_I(\tau_2)\rangle = S(\tau_1,\tau_2)|\psi_I(\tau_2)\rangle$$

The EOM:

$$-\frac{\partial}{\partial\tau}U(\tau) = V_I(\tau)U(\tau), \text{ or } -\frac{\partial}{\partial\tau}S(\tau_1,\tau_2) = V_I(\tau_1)S(\tau_1,\tau_2)$$

Solution is a time ordered exponential:

$$S(\tau_1,\tau_2) = T\exp\left[-\int_{\tau_1}^{\tau_2}V_I(\tau)d\tau\right]$$

We can use this to find perturbative expansion of Z:

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \mathrm{Tr}\left[e^{-\beta H_0}U(\beta)\right]$$

$$= \underbrace{\mathrm{Tr}\left[e^{-\beta H_0}\right]}_{Z_0}\cdot\underbrace{\frac{\mathrm{Tr}\left[e^{-\beta H_0}U(\beta)\right]}{\mathrm{Tr}\left[e^{-\beta H_0}\right]}}_{\langle U(\beta)\rangle_0} = Z_0\langle U(\beta)\rangle_0$$

which gives:

$$\frac{Z}{Z_0} = e^{-\beta\Delta F} = \left\langle T e^{-\int_0^\beta V_I(\tau)\, d\tau} \right\rangle$$

use only connected diagrams!

If we take the logarithm, we get the change in free-energy due to the interactions!

Green's functions

↳ for finite T, they are def. as:

$$\mathcal{G}_{\lambda,\lambda'}(\tau-\tau') = -\langle T\, \psi_\lambda(\tau)\, \psi^\dagger_{\lambda'}(\tau') \rangle$$

$$= -\mathrm{Tr}\left[e^{-\beta(H-F)}\, \psi_\lambda(\tau)\, \psi^\dagger_{\lambda'}(\tau') \right]$$

↳ bos, fer. field in Heisenberg rep.

$$F = -T \ln Z$$

$\mathcal{G}_{\lambda,\lambda'}$: function of $\tau-\tau'$ if H is time independent!

Often: look at diagonal Green's functions: $\mathcal{G}_{\lambda,\lambda'}(\tau) = \delta_{\lambda,\lambda'}\mathcal{G}_\lambda(\tau)$

For cont. variables: write $\mathcal{G}(\vec{p}, \tau)$

Free system:

$$H = \sum \varepsilon_\lambda \psi_\lambda^\dagger \psi_\lambda \qquad \varepsilon_\lambda = E_\lambda - \mu$$

$$\langle \psi_{\lambda'}^\dagger \psi_\lambda \rangle = \delta_{\lambda,\lambda'} \times \begin{cases} n(\varepsilon_\lambda) & \text{bos} \\ f(\varepsilon_\lambda) & \text{fem} \end{cases} \qquad n(\varepsilon_\lambda) = \frac{1}{e^{\beta\varepsilon_\lambda} - 1} \qquad f(\varepsilon_\lambda) = \frac{1}{e^{\beta\lambda} + 1}$$

$$\langle \psi_\lambda \psi_{\lambda'}^\dagger \rangle = \delta_{\lambda,\lambda'} \pm \langle \psi_{\lambda'}^\dagger \psi_\lambda \rangle$$

$$= \delta_{\lambda\lambda'} \begin{cases} 1 + n(\varepsilon_\lambda) & \text{bos} \\ 1 - f(\varepsilon_\lambda) & \text{fem} \end{cases}$$

Time evolution:

$$\psi_\lambda(\tau) = e^{-\varepsilon_\lambda \tau} \psi_\lambda(0) \qquad \psi_\lambda^\dagger(\tau) = e^{\varepsilon_\lambda \tau} \psi_\lambda^\dagger(0)$$

The Green's function becomes

$$\mathcal{G}_{\lambda\lambda'}(\tau - \tau') = -\Big[\theta(\tau - \tau') \langle \psi_\lambda(\tau) \psi_{\lambda'}^\dagger(\tau') \rangle$$

$$+ \underbrace{\zeta}_{\substack{+1 \text{ bos} \\ -1 \text{ fem}}} \theta(\tau' - \tau) \langle \psi_{\lambda'}^\dagger(\tau') \psi_\lambda(\tau) \rangle \Big] e^{-\varepsilon_\lambda(\tau - \tau')}$$

diag case

$$\mathcal{G}_\lambda(\tau) = -e^{-\varepsilon_\lambda \tau} \begin{cases} \left[(1 + n(\varepsilon_\lambda))\theta(\tau) + n(\varepsilon_\lambda)\theta(-\tau)\right] \\ \left[(1 + f(\varepsilon_\lambda))\theta(\tau) - f(\varepsilon_\lambda)\theta(-\tau)\right] \end{cases}$$

↑ structure same as $T=0$ $G_\lambda^t(t)$, evaluated at $t = -i\tau$

$$\mathcal{G}_\lambda(\tau) = -i\, G_\lambda^t(-i\tau)$$

For generic Green's functions, we have for any τ:

$$g_{\lambda\lambda'}(\tau+\beta) = \zeta\, g_{\lambda\lambda'}(\tau)\,;$$

use cyclic property of Tr: $-\beta < \tau < 0$:

$$g_{\lambda\lambda'}(\tau) = \zeta \langle \psi^\dagger_{\lambda'}(0)\, \psi_\lambda(\tau) \rangle$$

$$= \zeta\, \mathrm{Tr}\left[e^{-\beta(H-F)}\, \psi^\dagger_{\lambda'}\, e^{\tau H}\, \psi_\lambda\, e^{-\tau H} \right]$$

cancel

$$= \zeta\, \mathrm{Tr}\left[e^{-\beta(H-F)}\, e^{(\tau+\beta)H}\, \psi_\lambda\, e^{-(\tau+\beta)H}\, \psi^\dagger_{\lambda'} \right]$$

$$= \zeta\, \cancel{\zeta} \langle \psi_\lambda(\tau+\beta)\, \psi^\dagger_{\lambda'}(0) \rangle$$

$$= \zeta\, g_{\lambda\lambda'}(\tau+\beta)$$

So, we can take τ outside of $[-\beta, \beta]$ if we assume this periodicity

$\Rightarrow$ We can use 'Matsubara ~~freq~~ frequencies!

For bosons: $\nu_n = 2\pi k_B T\, n$

fermion $\omega_n = \pi(2n+1)\, k_B T$

This gives $e^{i\nu_n(\tau+\beta)} = e^{i\nu_n \tau}$

$e^{i\omega_n(\tau+\beta)} = -e^{i\omega_n \tau}$

So, we can decompose $g_{\lambda\lambda'}$ in 'modes':

$$g_{\lambda\lambda'}(\tau) = k_B T \sum_n g_{\lambda\lambda'}(i\nu_n)\, e^{-i\nu_n \tau} \qquad (\text{Bos})$$

$$= k_B T \sum_n g_{\lambda\lambda'}(i\omega_n)\, e^{-i\omega_n \tau} \qquad (\text{fer}),$$

or, inverse:

$$g_{\lambda\lambda'}(i\alpha_n) = \int_0^\beta d\tau\, g_{\lambda\lambda'}(\tau)\, e^{i \overset{\overset{\omega_n \text{ or } \nu_n}{\uparrow}}{\alpha_n} \tau}$$

For free fermions, we find:

$$g_\lambda(i\omega_n) = -\int_0^\beta d\tau\, \underbrace{\left[1 - f(\epsilon_\lambda)\right]}_{\frac{1}{1+e^{-\beta\epsilon_\lambda}}} e^{i\omega_n \tau - \epsilon_\lambda \tau}$$

$$\underset{(e^{i\omega_n\beta} = -1)}{\overset{\uparrow}{=}} \frac{1}{i\omega_n - \epsilon_\lambda}$$

For free bosons: $g_\lambda(i\nu_n) = \dfrac{1}{i\nu_n - \epsilon_\lambda}$

Same structure, ~~pure~~ statistics is in 'periodicity of img. time'.

Free fermion: same as $T=0$, w/ $\omega \to i\omega_n$, and remove $i\delta\, \mathrm{sgn}(\epsilon_\lambda)$

We can write $N(\mu)$, # of particles, in terms of $\mathcal{G}_\lambda(i\omega_n)$:

$$N_\lambda = \langle c_\lambda^\dagger c_\lambda \rangle = -\langle T\, c_\lambda(0^-)\, c_\lambda^\dagger(0) \rangle = \mathcal{G}_\lambda(0^-),$$

$$\mathcal{G}_\lambda(\tau) = T k_B \sum_{n} \mathcal{G}_\lambda(i\omega_n) e^{-i\omega_n \tau}$$

$$N(\mu) = \sum_\lambda N_\lambda = k_B T \sum_{\lambda, n} \mathcal{G}_\lambda(i\omega_n)\, e^{i\omega_n 0^+}$$

Find the free energy by $-\frac{\partial F}{\partial \mu} = N(\mu)$, so

$$F = -\int^{\mu} d\mu\, N(\mu) = -k_B T \sum_{\lambda, n} \int^{\mu} d\mu \frac{e^{i\omega_n 0^+}}{i\omega_n - E_\lambda + \mu}$$

$$= -k_B T \sum_{\lambda, n} \ln\left[i\omega_n - E_\lambda + \mu \right] e^{i\omega_n 0^+} + C$$

$$= -T k_B \sum_{\lambda, n} \ln \mathcal{G}_\lambda^{-1}(i\omega_n) e^{i\omega_n 0^+} + C.$$

So, we need to be able to sum over the Matsubara frequencies (will give $C=0$).

We use contour integral method!

$$f(z) = \frac{1}{e^{z\beta}+1} \quad \text{has pole of strength } -k_B T$$

at each $z = i\omega_n$:

$$f(i\omega_n + \delta) = \frac{1}{e^{i\omega_n\beta + \delta}+1} \sim -\frac{1}{\beta\delta} = \frac{-k_B T}{\delta}$$

So we can write, for generic F

$$k_B T \sum_n F(i\omega_n) = \int_C \frac{dz}{2\pi i} F(z) f(z)$$

C clockwise around the poles of $f(z)$.
(− sign!)

C can be deformed to go counterclockwise around poles, branchcuts of $F(z)$.

Take $\langle c^+_{\vec{p}\sigma} c_{\vec{p}\sigma} \rangle = \mathcal{G}(\vec{p}, 0^-) = T \sum_n \frac{e^{i\omega_n 0^+}}{i\omega_n - \varepsilon(\vec{p})}$

$F(z)$ is $F(z) = \frac{e^{z0^+}}{z - \varepsilon(p)}$ single pole at $z = \varepsilon(p)$

$$\langle n_{\vec{p}\sigma} \rangle = k_B T \sum_n \frac{e^{i\omega_n 0^+}}{i\omega_n - \varepsilon(p)} = \int_{C'} \frac{dz}{2\pi i} \frac{1}{z - \varepsilon(p)} e^{z0^+} f(z)$$

$\rightarrow f(\varepsilon_p)$ as expected!

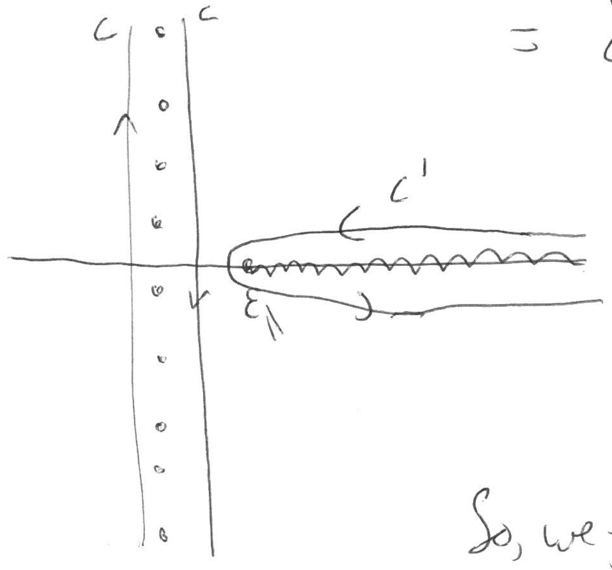
Now the Free energy:

$$F = -T \sum_{\lambda n} \ln(\epsilon_\lambda - i\omega_n)\, e^{i\omega_n 0^+} + C$$

Show that $F = -T \sum_{\lambda} \ln\left[1 + e^{-\beta \epsilon_\lambda}\right] + C$, so

C has to be zero.

$$F = \sum_{\lambda} \int_{C'} \frac{dz}{2\pi i} f(z) \ln(\epsilon_\lambda - z)\, e^{z 0^+} + C$$



$$= \sum_{\lambda} -\int_{\epsilon_\lambda}^{\infty} d\omega\, f(\omega) + C$$

$$= -k_B T \sum_{\lambda} \ln(1 + e^{-\beta\epsilon_\lambda}) + C$$

So, we find that $C = 0$ ~~for~~ to obtain the expected result.

Short recap of last week:

Finite Temperature: use imaginary time.

$$it \to \tau$$

full part. func.

$$\frac{Z}{Z_0} = \langle U(\beta) \rangle_0 = \mathrm{Tr}\left[T e^{-\int_0^\beta V_I(\tau) d\tau} \right]$$

free part. func.

We define Green's functions as:

$$\mathcal{G}_{\lambda,\lambda'}(\tau - \tau') = -\langle T \psi_\lambda(\tau) \psi^+_{\lambda'}(\tau') \rangle$$

(simple exp. for free bosons & fermions).

Periodicity: $-\beta < \tau < 0$: $\mathcal{G}_{\lambda\lambda'}(\tau + \beta) = \pm \mathcal{G}_{\lambda\lambda'}(\tau)$

↳ Used to go to Matsubara frequencies:

$$\nu_n = 2\pi n k_B T \quad \text{bosons} \qquad \left(e^{i\nu_n(\tau+\beta)} = e^{i\nu_n \tau} \right.$$

$$\omega_n = 2\pi(n+\tfrac{1}{2}) k_B T \quad \text{fermions} \qquad \left. e^{i\omega_n(\tau+\beta)} = -e^{i\nu_n \tau} \right)$$

Free bosons/fermions $\mathcal{G}_\lambda(i\alpha_n) = \frac{1}{i\alpha_n - \varepsilon_\lambda}$ $\quad \alpha_n = \begin{cases} \nu_n & \text{bos} \\ \omega_n & \text{fem} \end{cases}$

$$\left[\mathcal{G}_{\lambda,\lambda'}(\tau) = k_B T \sum_n \mathcal{G}_{\lambda\lambda'}(i\alpha_n) e^{-i\alpha_n \tau} \right]$$

$$\mathcal{G}_{\lambda\lambda'}(i\alpha_n) = \int_0^\beta d\tau \, \mathcal{G}_{\lambda\lambda'}(\tau) e^{i\alpha_n \tau}$$

For free fermions, $N(\mu)$ and F are given by

$$N(\mu) = k_B T \sum_{\lambda, n} \mathcal{G}_\lambda(i\omega_n)\, e^{i\omega_n 0^+};$$

$$F(T,\mu) = -k_B T \sum_{\lambda, n} \ln\left[-\mathcal{G}_\lambda^{-1}(i\omega_n)\right] e^{i\omega_n 0^+} + C(T)$$

will show this is zero!

Contour integral method: replace sum over Matsubara frequencies by integral.

Fermi-function: $f(z) = \frac{1}{e^{z\beta}+1}$ has a pole of strength $-1/\beta = -k_B T$ at each $z = i\omega_n$:

$$f(i\omega_n + \delta) = \frac{1}{e^{(i\omega_n+\delta)\beta}+1} \simeq -\frac{1}{\beta\delta}$$

So we can write

$$k_B T \sum_n F(i\omega_n) = \oint_{C} \frac{dz}{2\pi i} F(z) f(z)$$

where C goes clockwise (neg. direction) around poles of $f(z)$.

Q: ~~wh~~ how to do ~~the~~ bosonic case?

C: deformed to contour C' in pos direction around poles & branch-cuts of $F(z)$: use that ~~con~~ int. over large circle at ∞ gives zero!

particles (use mom. representation):

$$\langle c^+_{\vec{p}\sigma} c_{\vec{p}\sigma}\rangle = \mathcal{G}(\vec{p}, 0^-) = k_B T \sum_n \frac{e^{i\omega_n 0^+}}{i\omega_n - \varepsilon(\vec{p})}$$

So we take $F(z) = \dfrac{e^{z0^+}}{z - \varepsilon(\vec{p})}$, which has

a single pole at $z = \varepsilon(\vec{p})$

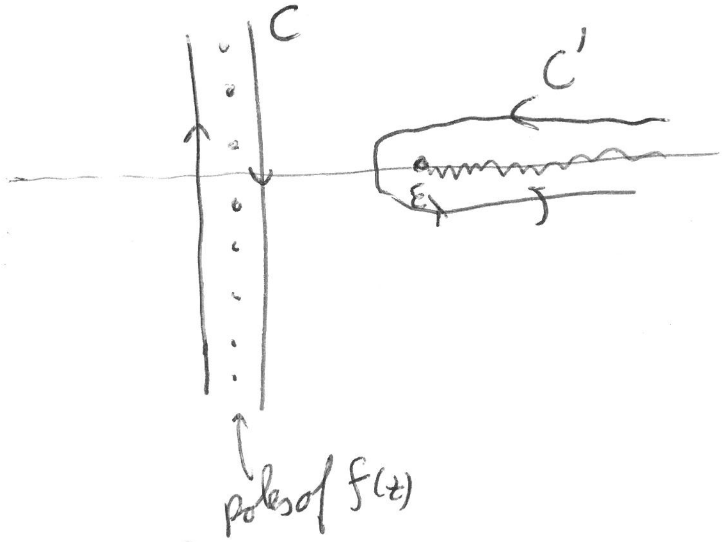
$$\langle n_{\vec{p}\sigma}\rangle = \int_{C'} \frac{dz}{2\pi i} \frac{1}{(z - \varepsilon(\vec{p}))} e^{z0^+} f(z) = f(\varepsilon(\vec{p}))$$

Look at the free-energy:

$$F = -k_B T \sum_{\lambda, n} \ln[\varepsilon_\lambda - i\omega_n]\, e^{i\omega_n 0^+} + C(T)$$

$$= \sum_\lambda \int_{C'} \frac{dz}{2\pi i} f(z) \ln(\varepsilon_\lambda - z)\, e^{z0^+} + C(T)$$

t



Due to branch cut: dif of log at $\omega + 0^+$ and $\omega + 0^-$: $2\pi i$

$$F = \sum_\lambda -\int_{\varepsilon_\lambda}^{\infty} d\omega\, f(\omega) + C(T) = -k_B T \sum_\lambda \ln(1 + e^{-\beta\varepsilon_\lambda}) + C(T)$$

So we conclude that $\mathcal{Q}D$ has to be zero!

Generating function & Wick: Free particles

$$H_0 = \sum_\lambda \varepsilon_\lambda \psi^\dagger_\lambda \psi_\lambda \quad , \quad V(\tau) = -\sum_\lambda \left[\bar{\eta}_\lambda(\tau) \psi_\lambda + \psi^\dagger_\lambda \eta(\tau) \right]$$

source term to free particle system.

Similarly as to $T=0$ case:

$$Z_0[\bar{\eta}, \eta] = Z_0 \left\langle T e^{-\int_0^\beta V_I(\tau) d\tau} \right\rangle_0$$

$$= Z_0 \left\langle T e^{+\int_0^\beta \sum_\lambda \left(\bar{\eta}_\lambda(\tau)\psi_\lambda(\tau) + \psi^\dagger_\lambda(\tau)\eta_\lambda(\tau) \right)} \right\rangle$$

$$\frac{Z_0[\bar{\eta}, \eta]}{Z_0} = e^{-\sum_\lambda \int_0^\beta d\tau_1 d\tau_2 \bar{\eta}_\lambda(\tau_1) g_\lambda(\tau_1 - \tau_2) \eta_\lambda(\tau_2)}$$

Where $g_\lambda(\tau_1 - \tau_2) = -\langle T \psi_\lambda(\tau_1) \psi^\dagger_\lambda(\tau_2) \rangle$ (see book!)

g can be obtained using functional derivatives, see book.

Again, we can use power series ~~exp.~~ expansion to find a Wick's theorem:

$$(-1)^n g(1,2,\dots,n; 1',2',\dots,n') = \langle T \psi(1) \cdots \psi(n) \psi^\dagger(n') \cdots \psi^\dagger(1') \rangle$$

$$g(1,\dots,n, n',\dots,1') = \sum_\sigma (-1)^\sigma_{\text{ferm}} \prod_{r=1}^{n} g(r - \sigma_r')$$

We do not know what the ~~F~~ $T>0$ vacuum is, ~~to hard to der~~ we can not derive this by commuting destruction ~~p~~ operators to act on the vacuum!

Feynman diagram expansion (brief!)

Have to perform expansion of $Z = e^{-\beta F}$ ~~F~~ where

$F = E - TS - \mu N$;

$Z = Z_0 \langle T e^{-\int_0^\beta V_I(\tau) d\tau} \rangle$

Works analogously to $T=0$ case, w/ few modifications

$Z = Z_0$ "$\sum$ all diagrams"

$\Delta F = F - F_0 = -k_B T \ln\left[\frac{Z}{Z_0}\right] = -k_B T$ "$\sum$ linked diagrams"

(linked cluster theorem).

~~Changes~~

~~$\Rightarrow$~~ ~~Tw~~ Green's function: $g(1-2) = \sum$ two leg diagrams

Changes: $-i \to -1$ in time ordered exponential

$\int_{-\infty}^{\infty} dt \to \int_0^\beta d\tau$

Use notation:

$$\int d1\, d2\ \bar{\eta}(1)\, g(1-2)\, \eta(2) = \bar{\eta} \leftarrow \eta$$

$d1$: integrate over $(\vec{x}_1, \tau_1) = (\vec{1}, \tau_1)$

Non-interacting gen. function:

$$\frac{Z_0[\bar{\eta}, \eta]}{Z_0} = \langle \hat{S} \rangle_0 = \exp[-\bar{\eta} \leftarrow \eta]$$

$$\hat{S} = T\exp\left[\int_0^\beta d1 \left[\bar{\eta}(1)\psi(1) + \psi^\dagger(2)\eta(2)\right]\right]$$

Functional derivatives w.r.t. source terms 'brings down operators':

$$\frac{\delta}{\delta\bar{\eta}(1)} \langle T \ldots \hat{S} \rangle_0 = \langle T \ldots \psi(1) \ldots \hat{S} \rangle_0$$

$$(\pm)\frac{\delta}{\delta\eta(2)} \langle T \ldots \hat{S} \rangle_0 = \langle T \ldots \psi^\dagger(2) \ldots \hat{S} \rangle_0$$

So, expectation value of operator $F[\psi^\dagger, \psi]$ is formally:
some poly.

$$\langle T\, F[\psi^\dagger, \psi]\, \hat{S} \rangle_0 = \left(F\left[\underset{(\pm)}{\zeta}\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] e^{(-\bar{\eta} \leftarrow \eta)}\right)$$

In particular:

$$\frac{Z[\bar{\eta}, \eta]}{Z_0} = \langle T e^{-\int_0^\beta \hat{V}(\tau)d\tau}\, \hat{S} \rangle_0$$

$$= \langle e^{-\int_0^\beta d\tau\, V(\zeta \frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar\eta})} e^{(-\bar\eta \leftarrow \eta)} \rangle_0$$

We can get rid of $(-)$ in exp. coming from Green's function:

$(\eta, \bar\eta) \to (\alpha, -\bar\alpha)$, but we'll get $(-1)^{n+1}$ in the other exponent!

If we use the expression for $\frac{Z[\bar\eta,\eta]}{Z_0}$ as starting point ~~of~~ to generate Feynman diagrams, we obtain:

$Z[\bar\alpha, \alpha] = Z_0 \sum \{\text{all diagrams}\}$.

For the Feynman rules: see book.

Linked cluster theorem:

For free energy we only should sum over the linked diagrams.

Use replica trick to get some feeling for this:

$$\ln\left(\frac{Z}{Z_0}\right) = \lim_{n\to 0} \frac{1}{n}\left[\left(\frac{Z}{Z_0}\right)^n - 1\right]$$

We can write term w/ $\left(Z/Z_0\right)^n$ as coming from n replica's:

$$\left(\frac{Z}{Z_0}\right)^n = \left\langle e^{-\int_0^\beta d\tau \sum_{\lambda=1}^{n} \hat{V}^{(\lambda)}(\tau)} \right\rangle_0$$

So, in diagram expansion, each diagram has a replica index that should be summed over:

Single unlinked diagram: order $O(n)$, k unlinked diagrams $O(n^k)$. So, in limit $n \to 0$, only linked diagrams ~~survive~~ survive

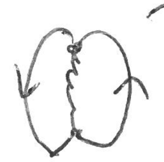
So: $-\beta \Delta F = \ln(Z/Z_0) = \sum$ closed, linked diagrams (real space).

Momentum space: obtain, after Fourier transform, an overall integral over center of mass coordinates, that contributes w/ factor $V\beta$.

Thus: $\frac{\Delta F}{V} = -\sum$ closed, linked diagram in mom. space.

Application: lowest order correction to the free energy of interacting fermi-gas:

Hartree-Fock:

$$\frac{\Delta F_{HF}}{V} = -\Big[$$  $+$  $$\Big]$$

Each loops: $-(2S+1)$

fermi, fermi sign, spin

$$\frac{\Delta F_{HF}}{V} = \underbrace{\frac{1}{2}}_{\text{symmetry factor}} \sum_{k} \mathcal{G}(k) \sum_{k'} \mathcal{G}(k') \left\{ \underbrace{\overset{-(2S+1)}{\cancel{(2S+1)}}\, V(k-k')}_{\text{(Fock)}} + \underbrace{[(2S+1)]^2 V(q=0)}_{\text{Hartree}} \right\}$$

$$\sum_{k} \mathcal{G}(k) = \int_{k} T \sum_{n} \frac{1}{i\omega_n - \varepsilon_k} e^{i\omega_n 0^+}$$

, so we can evaluate the sums, resulting in fermi-functions:

$$\frac{\Delta F_{HF}}{V} = \frac{1}{2} \int d\vec{k} \int d\vec{k}' \left[\underbrace{(2S+1)^2 V(q=0)}_{\substack{\text{Hartree cont.} \\ \infty \text{ for Coulomb, but} \\ \text{finite in practice, due} \\ \text{to neutralizing background!}}} - (2S+1) V(\vec{k}-\vec{k}') \right] f_{\vec{k}} f_{\vec{k}'}$$

$\parallel$ spins: wf antisym, so e^- further apart!

$\rho = (2S+1) \sum f_k$: classical interaction