Fermi liquids.

↳ stable phase of matter:
Coulomb energy & kin. energy are of same order, yet free theory captures alot already:
(metals)

* fermi-surface
* $c_v \propto \gamma T$
* χ constant.

Applicable: ~~A~~ * l. ^3He

* metals at low T.
* atomic systems (fermions in trap)
* neutron stars

Why can we do this:

* adiabatic principle!
* stable 'quasi-particles'

Landau's idea: excitations are ~~to~~ labelled by same quantum numbers as electrons/holes:
charge, spin, momentum.

They are called: quasi-particles and consist of many electrons.

At zero $T=0$, the distribution function remains $\Theta(p-p_F)$, so there is a FS at $T=0$

Quasi-particles have a finite lifetime:

~~One w/ $\vec{p}_1$ ar~~

Quasi-electron w/ $\vec{p}_1$ can decay:

$$|\vec{p}_1, -\rangle \to |\vec{p}_2, -\rangle + |p_3, -\rangle + |\vec{p}_1 - \vec{p}_2 - \vec{p}_3, +\rangle$$
$\qquad\qquad\quad$ qe $\qquad\quad$ qe $\qquad\qquad\qquad$ qh

But, when $|\vec{p}_1|$ becomes close to p_F, the phase-space becomes very small, $|p-p_F|^2 \to \tau_{qe} \sim |p-p_F|^{-2}$.

So quasi-particles are only defined close to Fermi-surface!

Validity: $T << T_F$.

$^3He \sim 7K$

$Na \sim 40000K$

Sommerfeld model: (non-interacting):

~~has~~ Density of states at ε_F is large:

Typical integrals: $2L^3 \int \frac{d^3k}{(2\pi)^3} f(\varepsilon_k) = \frac{V 8\pi}{8\pi^3} \int dk\, k^2 f(\varepsilon_k)$

$\cancel{\left(\frac{L}{2\pi}\right)^3}$ $\left(\frac{2\pi}{L}\right)^3$: volume of one $\vec{k}$ point $\qquad$ 2: spin

$$= \frac{V}{\pi^2} \int d\varepsilon\, k^2 \frac{dk}{d\varepsilon} f(\varepsilon_k)$$

$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ $\quad$ $k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

~~$\frac{d\varepsilon}{dk} = \frac{\hbar^2}{m} k =$~~

$N(\varepsilon) = \frac{k^2}{\pi^2} \frac{dk}{d\varepsilon}$ ~~$= \frac{2m\varepsilon}{\hbar^2}$~~

$k = \sqrt{\frac{2m\varepsilon_k}{\hbar^2}}$ $\qquad$ $\frac{dk}{d\varepsilon} = \frac{1}{2}\sqrt{\frac{2m}{\hbar^2 \varepsilon}}$

$$= \frac{2m\varepsilon}{2\hbar^2 \pi^2} \sqrt{\frac{2m}{\hbar^2} \frac{1}{\varepsilon}}$$

$$= \frac{m}{\hbar^3 \pi^2} \sqrt{2m\varepsilon} = \frac{m p_F}{\hbar^3 \pi^2}$$

$k = k_F$

Tot density of states per volume (including spin factor).

Energy: rel. to ~~#~~ chem. potential

$N(\varepsilon) \quad \varepsilon = E - \mu$

B-field: $\Delta E = -\underset{\pm 1}{\sigma} \mu_B B$

$\mu_B = \frac{g}{2}\frac{e\hbar}{2m}$

μ_B g factor: ≈ 2 for electron.

Diff in ↑ and ↓ particles:

$\delta N_\uparrow = -\delta N_\downarrow = \frac{1}{2} N(0)\, \mu_F B$

This gives magnetisation $M = \chi B$, w/ $\chi = \mu_F (N_\uparrow - N_\downarrow)/B$

$= \mu_F^2 N(0) = \mu_B^2 N(0)$

free electron gas.

C_V: Energy: $\mathcal{E}(T) = E - \mu N$

$$= \sum_{k,\sigma} \varepsilon_k \frac{1}{e^{\beta \varepsilon_k}+1}$$

$$C_V = \frac{d\mathcal{E}}{dT} = N_0 \int_{-\infty}^{\infty} d\varepsilon\, \varepsilon \frac{d}{dT}\left(\frac{1}{e^{\beta\varepsilon}+1}\right) = \underbrace{\frac{\pi^2}{3} N(0) k_B^2}_{\gamma} T$$

Quasi-particles: $\vec{p}, q, \sigma$ not changed, but other properties, mass, g-factor are 'renormalised'.

Described by a set of 'Landau-parameters'.

$n_{p,\sigma}$: # particles w/ $\vec{p}, \sigma$ in non-interacting system $(=0,1)$

Ψ_0: $(=1$ in GS. if $p<p_F)$

Ψ_0 evolves into interacting GS, with (unknown energy E_0).

Add fermion $p_0\sigma_0$ above FS, (non-interacting), and slowly turn on V again

$\Psi_{p_0\sigma_0}$: $n_{p\sigma} = \begin{cases} 1 & p<p_F \text{ or } p=p_0 \wedge \sigma=\sigma_0 \\ 0 & \text{other} \end{cases}$

Mom of state: $\vec{p}_0$ ($|\vec{p}_0| > p_F$), energy $E(p_0) > E_0$

Energy to create qp: $E^{(0)}_{p_0} = E(p_0) - E_0$

In grand can. ensemble $\mathcal{E} = E - \mu N$, to we get

$$\mathcal{E}^{(0)}_{p_0} = E^{(0)}_{p_0} - \mu = \mathcal{E}(p_0) - \mathcal{E}_0 \quad , \text{ positive energy}$$

Quasi-hole state:

$$\Psi_{p_0\sigma_0} : \quad n_{p\sigma} = \begin{cases} 1 & p < p_F \text{ but } p\sigma \neq p_0\sigma_0 \\ 0 & \text{o/w.} \end{cases}$$

$|p_0| < p_F$

Energy: $\bar{E}(p_0) = E_0 - E(p_0)$, or

$$\bar{\mathcal{E}}^{(0)}_{p_0} = -E^{(0)}_{p_0} + \mu = -\mathcal{E}^{(0)}_{p_0} \quad , \text{ this is positive, because}$$

$$\mathcal{E}^{(0)}_{p_0} < 0 \text{ for } p_0 < p_F$$

The qp are also well-def. if there is a finite density of them: $n_{\vec{p}\sigma}$ ~~are~~ ? do not change, because at FS, qp have a lifetm, given large amount of conserved quantities

The idea is now to write energy functional in terms of qp occupation numbers, and look at fluctuations $\delta n_{p\sigma} = n_{p\sigma} - n^{(0)}_{p\sigma}$:

$$\mathcal{E}(\{n_{p\sigma}\}) = E(\{n_{p\sigma}\}) - \mu N$$

$$\mathcal{E} = \mathcal{E}_0 + \sum_{\vec{p}\sigma} \left(E^0_{p\sigma} - \mu \right) \delta n_{\vec{p}\sigma}$$

$$+ \frac{1}{2} \sum_{\vec{p},\vec{p}',\sigma\sigma'} f_{p\sigma,p'\sigma'}\, \delta n_{p\sigma}\, \delta n_{p'\sigma'}$$

∘ $\varepsilon^{(0)}_{p\sigma} = \dfrac{\delta \mathcal{E}}{\delta n_{p\sigma}}$ energy of isolated excitation

W/o spin-orbit: $\varepsilon^0_{p\sigma} = \varepsilon^{(0)}_p - \sigma\mu_F B$

We can expand qp energy around fermi surface:

$$E^{(0)}_p = v_F (p - p_f) + \mu^{(0)}$$

Fermi energy, or chem pot in g.s.

Effective mass, in terms of v_F: $v_F = \left.\dfrac{d\varepsilon^{(0)}_p}{dp}\right|_{p_f} = \dfrac{p_f}{m^*}$

This gives the dos of the quasi-particles:

$$N^*(0) = \frac{m^* p_f}{\pi^2 \hbar^3}$$

Second order term: $f_{p\sigma,p'\sigma'} = \left.\dfrac{\delta^2 \mathcal{E}^*}{\delta n_{p\sigma}\, \delta n_{p'\sigma'}}\right|_{\text{other } n\text{'s const.}}$

Landau: m^* ~~relates to~~ due to interaction, di-polar part.

The quasi-energy: modified by the interaction,

$$\frac{\delta \mathcal{E}}{\delta n_{p\sigma}} = \varepsilon_{p\sigma} = E_{p\sigma} - \mu = \varepsilon^{(0)}_{p\sigma} + \sum_{p'\sigma'} f_{p\sigma,p'\sigma'}\, \delta n_{p'\sigma'}$$

~~For der~~ Total description, need to keep track of the entropy as well:

S: obtained by counting # of ~~sta~~ acc. states.

This is not changed by interactions, so can use free el. gas result:

$$S = -k_B \sum_{p,\sigma} (n_{\sigma p} \ln n_{p\sigma}) + (1-n_{p\sigma}) \ln (1-n_{p\sigma})$$

This gives the Free-energy:

$$F = \mathcal{E} - TS = E - \mu N - TS$$

$$F(n_{p\sigma}) = \mathcal{E}_0(\mu) + \sum_{p\sigma} \varepsilon^0_{p\sigma}\, \delta n_{p\sigma} + \frac{1}{2} \sum_{pp'\sigma\sigma'} f_{p\sigma p'\sigma'}\, \delta n_{p\sigma}\, \delta n_{p'\sigma'}$$

$$+ k_B T \sum_{p,\sigma} \left[n_{p\sigma} \ln n_{p\sigma} + (1-n_{p\sigma}) \ln (1-n_{p\sigma}) \right]$$

The $F_l^{s,a}$: Landau parameters, characterising the interactions

F_l^a: magnetic part.

OBS: Important that most physical quantities depend on $F_0^s, F_1^s, F_0^a, F_1^a$, to only a few parameters!

Note: $F_l^{s,a} = 2 \sum_{\vec{p}'} f^{s,a}_{\vec{p},\vec{p}'} P_l (\cos \theta_{\vec{p}\vec{p}'}) \delta(\varepsilon_{\vec{p}'})$

↳ useful for calculations (see book).

Important: form only valid for short-range interaction ($V(\vec{q}=\vec{0})$ should be finite): not true for Coulomb. Due to screening most goes through unchanged, however! (see sec. 6.6).

With interactions, equilibrium distribution of qp's is preserved. $\delta F = 0$ for small changes in $n_{\vec{p}\sigma}$

$$\delta F = \sum_{\vec{p},\sigma} \delta n_{\vec{p}\sigma} \left[\varepsilon_{\vec{p}\sigma} + k_B T \ln\left(\frac{n_{\vec{p}\sigma}}{1-n_{\vec{p}\sigma}} \right) \right] + \mathcal{O}(\delta n^2)$$

Leads to $n_{\vec{p}\sigma} = \dfrac{1}{e^{\beta \varepsilon_{\vec{p}\sigma}}+1} = f(\varepsilon_{\vec{p}\sigma})$: Fermi-Dirac

Parametrization of the interaction.

If spin is conserved, we have spin-rot. symmetry, sets the form of f's:

$$f_{p\sigma, p'\sigma'} = f^{s}_{p,p'} + f^{a}_{p,p'} \sigma\sigma'$$

magnetic comp. of the interac

[$n_{p\sigma}$'s: diagonal elements of density matrix $\rightsquigarrow$ $f_{p\alpha\beta, p'\gamma\delta} = f^{s}_{p,p'} \delta_{\alpha\beta}\delta_{\gamma\delta} + f^{a}_{pp'} \vec{\sigma}_{\alpha\beta} \cdot \vec{\sigma}_{\gamma\delta}$]

We are only interested in p's that are close to fermi-surface:

$$\vec{p} \approx \hat{p}\, p_f \quad ; \quad \vec{p}' = \hat{p}'\, p_f$$

If the FLiq. is isotropic: ~~can~~ only angle between $\vec{p}$ and $\vec{p}'$ matters: $\cos\theta = \hat{p} \cdot \hat{p}'$

We write: $f^{s,a}_{p,p'} = f^{s,a}(\cos\theta)$, or in dimensionless units:

$$F^{s,a}(\cos\theta) = N^{*}(0)\, f^{s,a}(\cos\theta)$$

These are expanded as:

$$F^{s,a}(\cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, F_l^{s,a}\, P_l(\cos\theta)$$

$\hookrightarrow$ Legendre poly's.

There is feedback effect from the interaction:

$$\varepsilon_{\vec{p},\sigma} = \varepsilon^0_{\vec{p}\sigma} + \sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma,\vec{p}'\sigma'} \, \delta n_{\vec{p}'\sigma'}$$

W/O any fields, $\delta n_{\vec{p}\sigma} \to 0$ if $T \to 0$, which gives

$$n_{\vec{p}\sigma} = f(\varepsilon^{(0)}_{\vec{p}}), \text{ giving } n_{\vec{p}\sigma} = \theta(-\varepsilon^{(0)}_p)$$
$$(T=0) \qquad = \theta(\mu - E^{(0)}_p)$$

For the specific heat:

$$C_V = \sum_{\vec{p}\sigma} \varepsilon^{(0)}_{\vec{p}\sigma} \frac{\partial f(\varepsilon^0_p)}{\partial T} \Rightarrow N^*(0) \int_{-\infty}^{\infty} d\varepsilon \, \varepsilon \frac{\partial f(\varepsilon)}{\partial T}$$

$$= \gamma T \text{ w/ } \gamma = \frac{\pi^2 k_B^2}{3} N^*(0)$$

Feedback effects of interactions.

FS: a deformable sphere that changes its shape when subjected to 'perturbing fields'.
- collective modes (not treated, see sec. 6.3).

Fields: chemical pot: $\delta\varepsilon^0_{\vec{p}\sigma} = -\delta\mu$

mag. field $\delta\varepsilon^0_{\vec{p}\sigma} = -\sigma\mu_F B$

'vector pot' (gives current) $\delta\varepsilon_{\vec{p}\sigma} = -\vec{A}\cdot\frac{e\vec{p}}{m}$

Coupling: to conserved quantity; not effected by interactions.

The change in $\varepsilon^0_{\vec{p}\sigma}$ due to field is not changed by the interactions.

There is, ~~of~~ however, feedback effect due to interactions:

$$\delta \varepsilon_{\vec{p}\sigma} = \delta \varepsilon^{(0)}_{\vec{p}\sigma} + \sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma,\vec{p}'\sigma'} \, \delta n_{\vec{p}'\sigma'}$$

Eq. particle occupation:

$$n_{\vec{p}\sigma} = f(\varepsilon^{(0)}_p + \delta\varepsilon_{\vec{p}\sigma}) = f(\varepsilon^{(0)}_p) + f'(\varepsilon^{(0)}_p)\,\delta\varepsilon_{\vec{p}\sigma}$$

For low T, 2^{nd} term becomes δ-function $-f'(\varepsilon) \sim \delta(\varepsilon)$

So we get $n_{\vec{p}\sigma} = \theta(-\varepsilon^{(0)}_p) + \underbrace{-\delta(\varepsilon^{0}_{p})(\delta\varepsilon_{\vec{p}\sigma})}_{\delta n_{\vec{p}\sigma}}$

$\delta n_{\vec{p}\sigma}$ is pol. of FS, which feeds back in the interaction

$$\delta\varepsilon_{\vec{p}\sigma} = \delta\varepsilon^{(0)}_{\vec{p}\sigma} + \sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma\vec{p}'\sigma'}\,\delta n_{\vec{p}'\sigma'}$$

$$= \delta\varepsilon^{(0)}_{\vec{p}\sigma} - \sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma\vec{p}'\sigma'}\,\delta(\varepsilon^{(0)}_{\vec{p}'})\,\delta\varepsilon_{\vec{p}'\sigma'}$$

Needs to be solved self-consistently

The symmetry of pert. is preserved, strength dep on Landau parameters.

charg. pert, $\vec{B}$: F_0^s and F_0^a,

motion via $\vec{A}$: dipolar $F_1^s \to m^*$

Bare change for multipole mode l:

$$\delta \varepsilon^{(0)}_{\vec{p}\sigma} = V_l \, Y_{lm}(\hat{p})$$

↳ spherical harm.

Renormalised response: changed magnitude:

$$\delta \varepsilon_{\vec{p}\sigma} = t_l \, Y_{lm}(\hat{p})$$

Feedback process gives contribution (Exercise!)

$$\sum_{\vec{p}'\sigma'} f_{\vec{p}\sigma \vec{p}'\sigma'} \, \delta n_{\vec{p}'\sigma'} = -F_l^s t_l \, Y_{lm}(\hat{p})$$

So in total $\delta \varepsilon_{\vec{p}\sigma} = (V_l - F_l^s t_l) \, Y_{lm}(\hat{p})$, giving

$$t_l = (V_l - F_l^s t_l), \text{ or } t_l = \frac{V_l}{1+F_l^s}$$

For repulsive int, $F_l^s > 0$, so a suppression!

For $F_l^s < 0$: enhancement. If -1, instability!

'a channel': similar result! $t_l^a = \dfrac{V_l^a}{1+F_l^a}$

Changes in susceptibilities: (isotropic pol. of FS).

$$\chi_c = \frac{1}{V}\frac{\partial N}{\partial \mu} \qquad \chi_s = \frac{1}{V}\frac{\partial M}{\partial B}$$

charge (or density):

Apply chem pot, or $\vec{B}$:

$$\delta \varepsilon^{(0)}_{\vec{p}\sigma} = \delta E^{0}_{\vec{p}\sigma} - \delta\mu = \sigma \mu_F B - \delta\mu, \text{ Full}$$

energies change as

$$\delta \varepsilon_{\vec{p}\sigma} = \sigma \lambda_s \mu_F B - \lambda_c \delta\mu \quad w/$$

$$\lambda_s = \frac{1}{1+F_0^a}; \quad \lambda_c = \frac{1}{1+F_0^s}, \text{ due to interactions}$$

$\delta\mu$: Or, $\delta N = \lambda_c N^*(0)\,\delta\mu$ so $\chi_c = \dfrac{N^*(0)}{1+F_0^s}$

δB: $\vec{B}$-field $\delta n_\uparrow = -\delta n_\downarrow = \lambda_s N^*(0)\mu_F B/2$, giving

$$\delta M = \mu_F(\delta n_\uparrow - \delta n_\downarrow) = \lambda_s \mu_F^2 N^*(0) B, \text{ or}$$

$$\chi_s = \frac{\mu_F^2 N^*(0)}{1+F_0^a}$$

Mass renormalisation

↳ due to dipolar component of interactions: F_1^S.

~~Partic~~ Fermions move through fluid, backflow enhance the mass: $m^* = m(1+F_1^S)$

This increases density of states, increase in C_V.

Can be seen through 'current': $v_F = \dfrac{p_F}{m^*} = \dfrac{p_F}{m} - \dfrac{p_F}{m}\left(\dfrac{F_1^S}{1+F_1^S}\right)$ backflow.

View backflow as response to dipolar distortion of FS when current is present.

We introduce 'fictitious' vector potential:

$q\vec{A} = A_N$ $(q=1)$, even though we look at neutral fluid.

H is then: $H[A_N] = \sum_\sigma \int d^3x \frac{1}{2m} \psi_\sigma^\dagger (-i\hbar\vec{\nabla} - \vec{A}_N)^2 \psi_\sigma(x) + \hat{V}$

[A_N: changes ~~p.~~ momentum of particles by $-\vec{A}_N$, so we can view the effect as ~~due to~~ fluid moving in ~~the~~ Galilean ref. frame w/ $u = A_N/m$.]

$\vec{A}_N$ couples to ~~conser.~~ conserved quantity: $\delta E = -\sum_{\vec{p}\sigma} \left(\frac{\vec{p}}{m} \cdot \vec{A}_N\right) n_{\vec{p}\sigma}$

So, change in energy w/o backflow ~~sam~~ in interacting case is the same:

$$\varepsilon^{0}_{\vec{p}\sigma} \to \varepsilon^{(0)}_{\vec{p}\sigma} + \delta\varepsilon^{(0)}_{\vec{p}\sigma}, \quad \delta\varepsilon^{(0)}_{\vec{p}\sigma} = -\frac{\vec{p}\cdot\vec{A}_N}{m} = -A_N \frac{p_F}{m}\cos\theta$$

where m is bare mass! Also get dipolar distortion & change in quasi-particle energy: $E_{\vec{p}-\vec{A}_N} = \frac{(p-A_N)^2}{2m^*}$

$$\delta\varepsilon_{\vec{p}\sigma} = -\frac{\vec{p}}{m^*}\cdot\vec{A}_N = -A_N \frac{p_F}{m^*}\cos\theta, \text{ with backflow!}$$

So, m^* appears!

Dipolar nature gives $\delta\varepsilon_{\vec{p}\sigma} = \left(\frac{1}{1+F_1^s}\right)\delta\varepsilon^{(0)}_{\vec{p}\sigma}$, or

$$1 + F_1^s = m^*/m$$

Density of states:

Typical integrals:

$$2\,\frac{L^3}{(2\pi)^3}\int d^3k\; f(\varepsilon_k)$$

$$= \frac{V\,8\pi}{\cancel{8}\pi^3}\int dk\; k^2 f(\varepsilon_k)$$

$$= \frac{V}{\pi^2}\int d\varepsilon\; k^2\,\frac{dk}{d\varepsilon}\, f(\varepsilon_k)$$

$$N(\varepsilon) = \frac{k^2}{\pi^2}\,\frac{dk}{d\varepsilon}$$: density of states per volume.

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\cancel{2}\, m\,\varepsilon}{\hbar^2\pi^2}\sqrt{\frac{2m}{\hbar^2}}\,\frac{1}{\cancel{2}\sqrt{\varepsilon}} \qquad k = \frac{1}{\hbar}\sqrt{2m\varepsilon} \qquad \frac{dk}{d\varepsilon} = \sqrt{\frac{2m}{\hbar^2}}\,\frac{1}{2\sqrt{\varepsilon}}$$

$$\underset{\text{at } \varepsilon_F}{=} \frac{m}{\hbar^3\pi^2}\sqrt{2m\varepsilon} = \frac{m k_F}{\hbar^2\pi^2} = \frac{m p_F}{\hbar^3\pi^2}$$