Real space version: $\psi_\sigma(x,t) = \int dk\, \psi_{k\sigma}\, e^{i(k\cdot x)}$

gives

$$g(x-x', t-t') = \int \frac{d^3k}{(2\pi)^3}\, g(k, t-t')\, e^{ik(x-x')}$$

Also:

$$g(k,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \underbrace{g(k,\omega)}_{\text{propagator.}}\, e^{-i\omega t}$$

---

Free fermion case: (w/ heat baths)

$$H = H_0 - \mu N = \underbrace{\sum_{\sigma,k}}_{\text{gives factor}} \varepsilon_k\, c^+_{k\sigma} c_{k\sigma}$$

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} - \mu$$

GS: $|\phi\rangle = \prod_{\sigma, |k|\le k_F} c^+_{k\sigma} |0\rangle$

Use Heisenberg rep: $c^+_{k\sigma}(t) = e^{i\varepsilon_k t}\, c^+_{k\sigma}$

$c_{k\sigma}(t) = e^{-i\varepsilon_k t}\, c_{k\sigma}$

'Forward propagation': $\langle\phi|\, c_{\sigma k}(t)\, c^+_{k'\sigma'}(t')\, |\phi\rangle = \delta_{\sigma\sigma'}\delta_{kk'}\, e^{-i\varepsilon_k(t-t')} \times \langle\phi| c_{k\sigma} c^+_{k\sigma} |\phi\rangle$

$$= \delta_{\sigma,\sigma'}\delta_{k,k'}\, (1 - \underbrace{n_{k\sigma}}_{=\theta(|k_F| - |\vec{k}|)})\, e^{-i\varepsilon_k(t-t')}$$

Backwards propagation: (destroy fermion, to create a hole):

$$\langle \phi \mid c^{\dagger}_{k,\sigma'}(t') \, c_{k,\sigma}(t) \mid \phi \rangle = \delta_{\sigma,\sigma'} \, \delta_{k,k'} \, n_k \, e^{-i\varepsilon_k (t-t')}$$

$$g(k, t) = -i \langle \phi | T(\psi_{k\sigma}(t) \, \psi^{\dagger}_{k\sigma}(0)) | \phi \rangle \qquad (t' = 0)$$

$$g(k,t) = -i \left[ (1-n_k)\,\theta(t) - n_k\,\theta(-t) \right] e^{-i\varepsilon_k t}$$

$$= \begin{cases} -i\,\theta_{|\vec{k}|-|k_F|} \, e^{-i\varepsilon_k t} & t > 0 \quad \text{particles} \\ i\,\theta_{|k_F|-|k|} \, e^{-i\varepsilon_k t} & t < 0 \quad \text{holes moving backward.} \end{cases}$$

To find propagator, we FT, w/ convergence fact.

$$g(k,\omega) = -i \int_{-\infty}^{\infty} dt \; e^{i(\omega-\varepsilon_k)t} \, e^{-|t|\delta} \left[ \theta_{k-k_F}\,\theta(t) - \theta_{k_F-k}\,\theta(-t) \right]$$

$$= -i \left[ \frac{\theta_{k-k_F}}{\delta - i(\omega-\varepsilon_k)} - \frac{\theta_{k_F-k}}{\delta + i(\omega-\varepsilon_k)} \right] = \frac{1}{\omega - \varepsilon_k + i\delta_k}$$

$\hookrightarrow \delta \operatorname{sing}(k-k_F)$

We can use GF to calculate ~~en~~ density, kin energy, etc.

$$\langle \hat{\rho}(x) \rangle = \sum_\sigma \langle \psi^{\dagger}_\sigma \psi_\sigma \rangle = -\sum_\sigma \langle \phi | T[\psi_\sigma(x,0^-)\,\psi^{\dagger}(x(0))] | \phi \rangle$$

$$= -i \underset{\substack{\uparrow \\ \text{spin } \uparrow,\downarrow}}{2} \, g(x, 0^-) \Big|_{x=0},$$

$$\langle \hat{T}(x) \rangle = \cdots = 2i \left[ \frac{\hbar^2 \nabla^2}{2m} \, g(x,0^-) \right]_{x=0}.$$

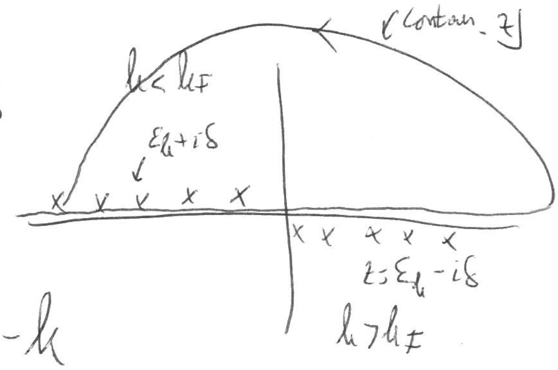
Gives relation $\langle T(x) \rangle = \frac{3}{5} \varepsilon_F \, \rho$

$$\langle \rho(x)\rangle = \overset{(\text{spin})}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi i}\, e^{+i\omega\delta}\,\frac{1}{\omega-\varepsilon_k+i\delta_k}$$

$$\langle \rho(x)\rangle = -i\,2\,\mathcal{G}(\vec{0}, 0^-)$$

$t = -\delta$

$\delta > 0.$




$$\int\frac{d\omega}{2\pi i}\, e^{i\omega\delta}\,\frac{1}{\omega-\varepsilon_k+i\delta_k} = \theta_{k_F-k}$$

only contributes if $k<k_F$

$$\rho = 2\int_{k<k_F}\frac{d^3k}{(2\pi)^3} = 2\cdot\frac{V_F}{(2\pi)^3} \qquad V_F = \frac{4}{3}\pi k_F^3$$

$$\langle T(x)\rangle = 2\int_{k<k_F}\frac{d^3k}{(2\pi)^3}\,\frac{\hbar^2 k^2}{2m} = \frac{3}{5}\,\varepsilon_F\,\rho \qquad \varepsilon_F = \frac{\hbar k_F^2}{2m}$$

---

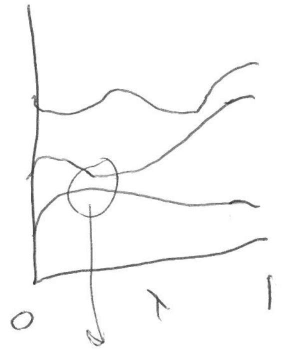
Adiabatic principle:

How can we hope to learn something about interacting systems?

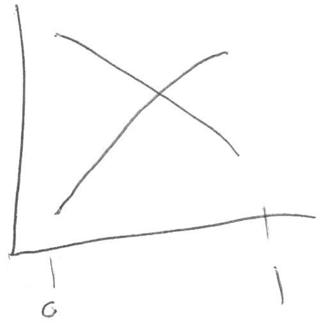
Take non-interacting system, slowly turn on interactions: $H = H_0 + \lambda(t)\,V$

$\hookrightarrow \lambda = e^{-|t|\delta}$ $\quad \delta \sim 1/T_A$

(Ground) states evolve. Two cases:




level repulsion
(levels have <u>same</u> symmetry)




levels cross:
↳ <u>different</u> symmetry!

spont. sym. breaking

Use $V(t) = e^{-|t|\varepsilon} V(0)$, so we turn $V$ on and off gradually. (take $\varepsilon \to 0$ limit at end).

At $t = -\infty$: $|-\infty\rangle$ is GS, both for Heisenberg & Interaction picture.

$$|\phi\rangle_H = |-\infty\rangle$$

Info about $V$ is in operators in Heisenberg picture

We are interested in Green's functions or correlators of interacting system. They can be written in terms of Green's function of non-interacting system at $t = -\infty$:

Gell-Mann / Low theorem.

$$\langle \phi | T A(t_1) B(t_2) \cdots N(t_n) | \phi \rangle_H$$

$$= \langle +\infty | T S(\infty,-\infty) A(t_1) \cdots N(t_n) | -\infty \rangle_I$$

where $S(\infty,-\infty) \equiv T\left[e^{-i\int_{-\infty}^{\infty} V(t')dt'}\right]$

All the info ~~ab~~ about interaction is encoded via the S-matrix!

We can use this as starting point for perturbation theory.

Sketch of proof (see book):

Write $|\psi(t)\rangle_I = U_I(t) |\psi\rangle_H$ $\quad |\psi\rangle_H = |-\infty\rangle$

$\hookrightarrow$ gs at any time!

Use rel. between $A_H(t) = U_I^\dagger(t) A_I(t) U_I(t)$

$S(t_1,t_2) = U_I(t_1,t_2) = U(t_1) U^\dagger(t_2)$ (assume $t_1 > t_2 \cdots > t_n$)

works for any time ordering!)

Gives $\langle \phi | A(t_1) \cdots N(t_n) | \phi \rangle_H$

$$= \underbrace{\langle -\infty | U_I^\dagger(t_1)}_{\langle -\infty | S^\dagger(t_1,-\infty) = \langle \infty | S(\infty, t_1)} A_I(t_1) S(t_1,t_2) B_I(t) \cdots S(t_{n-1},t_n) N(t_n) \underbrace{U(t_n)}_{S(t_n,-\infty)} |-\infty\rangle$$

$$= \langle -\infty | T S(\infty,-\infty) A_I(t_1) \cdots N_I(t_n) | -\infty \rangle$$

$\uparrow$

collect all S's using time ordering

Structure of Green's function in interacting case.

Pole in GF splits in many poles, with total strength 1:

$$g(k,\omega) = \sum_\lambda \frac{|M_\lambda(k)|^2}{\omega - \varepsilon_\lambda + i\delta_\lambda},$$

where $\sum_\lambda |M_\lambda(k)|^2 = 1$. GS $|\phi\rangle$: N particles

$$M_\lambda(k) = \begin{cases} \langle \lambda | c^+_{k\sigma} | \phi\rangle & : |\lambda\rangle \in |N+1\rangle \\ \langle \lambda | c_{k\sigma} | \phi\rangle & ; |\lambda\rangle \in |N-1\rangle \end{cases}$$

$E_\lambda - E_g > 0$ (always), and we have

$$\varepsilon_\lambda = \begin{cases} E_\lambda - E_g > 0 & |\lambda\rangle \in |N+1\rangle \\ -(E_\lambda - E_g) < 0 & |\lambda\rangle \in |N-1\rangle \end{cases} \qquad \delta_\lambda = \delta \,\mathrm{sgn}(\varepsilon_\lambda)$$

$g(k,\omega)$: same structure as in non-interacting case.

To show this: assume we know spectral decomposition of the interacting problem! $\sum_\lambda |\lambda\rangle\langle\lambda| = \mathbb{1}$

$$g(k,t) = -i\sum_\lambda \Big[ \langle \phi | c_{k\sigma}(t) \overset{|\lambda\rangle\langle\lambda|}{\phantom{x}} c^+_{k\sigma}(0) | \phi\rangle \theta(t) - \langle\phi| c^+_{k\sigma}(0) |\lambda\rangle\langle\lambda| c_{k\sigma}(t) |\phi\rangle \theta(-t) \Big]$$

$$\langle \phi | c_{k\sigma}(t) |\lambda\rangle = \langle\phi | c_{k\sigma} |\lambda\rangle e^{-i(E_\lambda - E_g)t} \quad \text{etc.}$$

So:

$$G(k,t) = -i \sum_{\lambda} \Big[ |\langle \lambda | c^{+}_{k\sigma} | \phi \rangle|^2 e^{-i(E_\lambda - E_g)t} \, \theta(t)$$

$$- |\langle \lambda | c_{k\sigma} | \phi \rangle|^2 e^{-i(E_g - E_\lambda)t} \, \theta(-t) \Big]$$

Doing FT:

$$G(k,\omega) = \sum_{\lambda} \left[ \frac{|\langle \lambda | c^{+}_{k\sigma} | \phi \rangle|^2}{\omega - (E_\lambda - E_g) + i\delta} + \frac{|\langle \lambda | c_{k\sigma} | \phi \rangle|}{\omega - (E_g - E_\lambda) - i\delta} \right]$$

One can show that

$$\sum_{\lambda} |M_\lambda(k)|^2 = \sum_{\lambda} |\langle \lambda | c^{+}_{k\sigma} | \phi \rangle|^2 + |\langle \lambda | c_{k\sigma} | \phi \rangle|^2$$

$$= 1$$

$$g(k,\omega) = -i \int_{-\infty}^{\infty} dt\, e^{i(\omega-\varepsilon_k)t}\, e^{-|t|/\delta} \Big[ \theta_{k-k_F}\, \theta(t) - \theta_{k_F-k}\, \theta(-t) \Big]$$

$$= -i \int_0^{\infty} dt\, e^{i(\omega-\varepsilon_k)t - t\delta}\, \theta_{k-k_F}$$

$$+ i \int_{-\infty}^{0} dt\, e^{i(\omega-\varepsilon_k)t + t\delta}\, \theta_{k_F-k}$$

$$= -i \left[ \frac{-1}{i(\omega-\varepsilon_k)-\delta}\, \theta_{k-k_F} \right]$$

$$+ i \left[ \frac{+1\, \theta_{k_F-k}}{i(\omega-\varepsilon_k)+\delta} \right] = \frac{\theta_{k-k_F}}{(\omega-\varepsilon_k)+i\delta} + \frac{\theta_{k_F-k}}{(\omega-\varepsilon_k)-i\delta}$$

$$= \frac{1}{(\omega-\varepsilon_k)+i\delta_k} \qquad \delta_k = \delta\, \mathrm{sgn}(k-k_F)$$

---

$$g(k,t) = -i \Big[ \langle\phi|\, c_{k\sigma}(t)\, c^{+}_{k\sigma}(0)\, |\phi\rangle\, \theta(t) - \langle\phi|\, c^{+}_{k\sigma}(0)\, c_{k\sigma}(t) |\phi\rangle\, \theta(-t) \Big]$$

$$= -i \sum_{\lambda} \Big[ \langle\phi|\, c_{k\sigma}(t) |\lambda\rangle \langle\lambda|\, c^{+}_{k\sigma}(0)\, |\phi\rangle\, \theta(t) - \langle\phi|\, c^{+}_{k\sigma}(0)\, \cancel{c_{k\sigma}(t)|\phi\rangle}\, |\lambda\rangle \langle\lambda| c_{k\sigma}(t)\, |\phi\rangle\, \theta(-t)$$

$$\langle\phi|\, c_{k\sigma}(t)\, |\lambda\rangle = \langle\phi| c_{k\sigma} |\lambda\rangle\, e^{i(E_g-E_\lambda)t}$$

$$\langle\lambda|\, c_{k\sigma}(t)\, |\phi\rangle = \langle\lambda|\, c_{k\sigma} |\phi\rangle\, e^{i(E_\lambda-E_g)t}$$

$$g(k,t) = -i \sum_\lambda \Big[ |\langle \lambda | c^\dagger_{k\sigma} | \phi \rangle|^2 e^{-i(E_\lambda - E_g)t}\, \theta(t)$$
$$- |\langle \lambda | c_{k\sigma} | \phi \rangle|^2 e^{-i(E_g - E_\lambda)t}\, \theta(-t) \Big]$$

Same as for free fer, if w/ $\varepsilon_k = E_\lambda - E_g$ first term

$\varepsilon_k = E_g - E_\lambda$ second term

$$g(k,\omega) = \sum_\lambda \left[ \frac{|\langle \lambda | c^\dagger_{k\sigma} | \phi \rangle|^2}{\omega - (E_\lambda - E_g) + i\delta} + \frac{|\langle \lambda | c_{k\sigma} | \phi \rangle|^2}{\omega - (E_g - E_\lambda) - i\delta} \right]$$