Interaction picture

H takes the form: $H = H_0 + V$, $V = \lambda V'$
with λ small.

Schrödinger picture: states evolve
Heisenberg " : operators "
Interaction " : states evolve w/o $e^{-iH_0 t/\hbar}$

So: $|\psi(t)\rangle_I = e^{+iH_0 t/\hbar} |\psi(t)\rangle_S$

$O_I(t) = e^{iH_0 t/\hbar} O_S \, e^{-iH_0 t/\hbar}$

Evolution of the states:

$|\psi(t)\rangle_I = U_I(t) \, |\psi(0)\rangle_I$

$\hookrightarrow U_I(t) = e^{iH_0 t/\hbar} e^{-iHt/\hbar}$

Arbitrary times:

$|\psi(t)\rangle_I = \underbrace{U(t) \, U^\dagger(t')}_{:= S(t,t')} |\psi(t')\rangle_I$

'Schrödinger eq': $i\hbar \partial_t |\psi(t)\rangle_I = \ldots = V_I(t) \, |\psi(t)\rangle_I$ (*)

where $V_I(t) = e^{iH_0 t/\hbar} \, V_S(t) \, e^{-iH_0 t/\hbar}$

We would like to 'exponentiate' (*) but this is non-trivial, $V(t)$ does not commute for different times! (so $e^{iV(t_1)+iV(t_2)\,t/\hbar} \neq e^{iV_1(t)\,t/\hbar}\,e^{iV_2(t)\,t/\hbar}$)

Solution: use time ordered exponential:

$$T[a(t_1)a(t_2)] = \begin{cases} a(t_1)a(t_2) & t_1 > t_2 \\ a(t_2)a(t_1) & t_2 > t_1 \end{cases}$$

The solution for $S(t,t')$ is:

$$S(t_2,t_1) = T\left[e^{-i/\hbar \int_{t_1}^{t_2} dt\, V(t)\, dt}\right]$$

See: book.

Or: write: $i\hbar \dfrac{\partial S(t_2,t_1)}{\partial t_2} = V_I(t_2)\, S(t_2,t_1)$

Equivalent to integral eqn:

$$* \begin{cases} \theta(t) = 0 & t<0 \\ \theta(t) = 1 & t>0 \end{cases}$$

$$S(t_2,t_1) = 1 - i/\hbar \int_{t_1}^{t_2} dt\; V_I(t)\, S(t,t_1)$$

$$= 1 - i/\hbar \int_{t_1}^{t_2} d\tau_1\, V_I(\tau_1) + \left(-i/\hbar\right)^2 \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2\, V_I(\tau_1)\, V(\tau_2)$$

solve by iteration

$+ \cdots$

Using $T[V(t_1)\cdots V(t_N)] = \sum_{\sigma} \theta(\tau_{\sigma_1}-\tau_{\sigma_2})\,\theta(\tau_{\sigma_2}-\tau_{\sigma_3})\cdots$
$V(\tau_{\sigma_1})\,V(\tau_{\sigma_2})\cdots V(\tau_{\sigma_N})$

Sum over all permutations.

Integrate this over all τ_i, from t_1 to t_2: all the $n!$ term give term we need, so we get:

$$S(t_2,t_1) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\frac{i}{\hbar}\right]^n \int_{t_1}^{t_2} d\tau_1 \cdots \int_{t_1}^{t_2} d\tau_n \, T\left[V_1(\tau_1) \cdots V_n(\tau_n)\right]$$

Exp: Driven harmonic oscillator

$$H_0 = \omega(b^\dagger b + \tfrac{1}{2}) \qquad V(t) = \bar{z}(t)\, b + b^\dagger z(t)$$

We assume: no perturbation at $t = \pm\infty$.

What is amplitude to stay in GS, if $V(t)$ is present?

$$S[\bar{z}, z] = \langle 0 | \, T e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \left[\bar{z}(t)\, b(t) + b^\dagger(t)\, z(t)\right]} |0\rangle,$$

where $b(t) = b e^{-i\omega t}$

How to evaluate: divide integral in 'time-slices', (à la path-integral!), and use commutators to normal order. ($b^\dagger$ to left, b's to right).

See book for ~~a~~ the calculation (or later on);

$$S[\bar{z}, z] = e^{-i/\hbar \int_{-\infty}^{\infty} dt\, dt'\, \bar{z}(t)\, \mathcal{G}(t-t')\, z(t')},$$

where $\mathcal{G}(t-t') = -i\theta(t-t')\, e^{-i\omega(t-t')}$:

Green's function

$S[\bar{z}, z]$ is a <u>functional</u> (Generating function)

$$= \langle 0| T\, e^{-i/\hbar \int_{-\infty}^{\infty} dt\, [\bar{z}(t) b(t) + b^{+}(t) z(t)]} |0) \qquad (*)$$

$$= e^{-i/\hbar \int_{-\infty}^{\infty} dt\, dt'\, \bar{z}(t)\, \mathcal{G}(t-t')\, z(t)}$$

Where $\mathcal{G}(t-t') = -i \langle 0| T\, b(t)\, b^{+}(t) |0)$

$\therefore$ use this to evaluate $\mathcal{G}(t-t')$; much easier!

To see this, ~~and~~ expand $(*)$ to 1^{st} order in $\bar{z}$ and z, and compare coefficients.

We can express $\mathcal{G}(t_1 - t_1')$ as

$$i^2 \left.\frac{\delta^2 S}{\delta z(t_1')\, \delta \bar{z}(t_1)}\right|_{\bar{z}z=0} = \langle 0| T\, b(t_1')\, b^{+}(t_1) |0) = i\,\mathcal{G}(t_1 - t_1'),$$

and similar for higher order ~~a~~ correlators:

$$i^{2n} \frac{\delta^{2n} S[\bar{z},z]}{\delta z(t_n') \cdots \delta z(t_1') \delta \bar{z}(t_n) \cdots \delta \bar{z}(t_1)}\Bigg|_{z=\bar{z}=0} = \langle 0| T b(1) \cdots b(n) b^\dagger(t_1') \cdots b^\dagger(t_n')|0\rangle$$

$$:= + i^n g(t_1, \ldots, t_n, t_1' \ldots, t_n')$$

To evaluate this, we first do the der. wrt $\bar{z}(t_i)$:

$$i^n \frac{\delta^n S}{\delta \bar{z}(t_n) \cdots \delta \bar{z}(t_1)} = S[\bar{z},z] \int ds_1' \cdots ds_n' \prod_i g(s_i - s_i') z(s_i')$$

For the diff. ~~wrt~~ wrt $z(t_i')$, there are $n!$ contribution (note, we set $z = \bar{z} = 0$ at the end!), so we need to diff, not $S[\bar{z},z]$).

So we get: $g(t_1, \ldots, t_n, t_1', \ldots, t_n') = \sum_\sigma \prod_{i=1}^{n} g(t_i - t'_{\sigma_i})$,

~~to~~ that is, the product over one-particle green's functions!

Quite remarkable result! (only non interacting systems!).

Wick's theorem!

$$J[f] = \int dx\, L(x, f(x), f'(x))$$

$$\delta J = \int dx \frac{\delta J}{\delta f} \delta f(x)$$

$$\frac{\delta J}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'}$$

Other formulation of Wick's theorem:
Relation between Normal and time-ordering:

$$T\left[\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\right]$$

$$= N\left[\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\right]$$

$$+ \langle 0|T\left[\varphi(x_1)\varphi(x_2)\right]|0\rangle \; N\left[\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_N)\right]$$

$$+\text{perm}$$

$$+ \langle 0|T\left[\varphi(x_1)\varphi(x_2)\right]|0\rangle \langle 0|T[\varphi(x_3)\varphi(x_4)]|0\rangle \;\times$$

$$\times\, N\left[\hat{\varphi}(x_5)\cdots\hat{\varphi}(x_N)\right]$$

$$+\text{perm}$$

$$\vdots$$

We assume commutators are c-numbers: $\left[\hat{\varphi}(x) = \varphi^{+}(x) + \varphi^{(-)}(x)\right]$

$$T\left[\varphi(x_1)\varphi(x_2)\right] = N\left[\varphi(x_1)\varphi(x_2)\right] + \text{c-number}$$

VEV: $\langle 0|T\left[\varphi(x_1)\varphi(x_2)\right]|0\rangle = 0 + \text{c-number},$

So $T\left[\varphi(x_1)\varphi(x_2)\right] = N\left[\varphi(x_1)\varphi(x_2)\right] + \langle 0|T(\varphi(x_1)\varphi(x_2))|0\rangle$

Can use this + induction.

Better way: use generating function form:

$J(x)$: c-number.

Then (*) $T\left\{ e^{-i\int d^4x\, J(x)\, \hat{\varphi}(x)} \right\}$

$$= N\left[e^{-i\int d^4x\, J(x)\, \hat{\varphi}(x)} \right] e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x) \langle 0|T\,\varphi(x)\varphi(y)|0\rangle J(y)}$$

Set $J(x) = i\,K(x)$, and expand (*) to get previous form of Wick's thm.

[Proof of (*) Maybe exercise?].

~~Note:~~ T Greens functions important in many-body systems.

$$\mathcal{G}_{\lambda\lambda'}(t-t') = -i \langle \phi | T\, \psi_\lambda(t)\, \psi^\dagger_{\lambda'}(t') | \phi \rangle$$

Many body GS.

$$T\left[\psi_\lambda(t)\, \psi^\dagger_{\lambda'}(t')\right] = \begin{cases} \psi_\lambda(t)\, \psi^\dagger_{\lambda'}(t') & t > t' \\ \pm\, \psi^\dagger_{\lambda'}(t')\, \psi_\lambda(t) & t' > t \end{cases}$$

bos / fermions

$\mathcal{G}_{\lambda\lambda'}(t-t')$: $\lambda, t \longleftarrow \lambda', t'$

Translation & ~~spin~~ conserved spin:

$$\mathcal{G}_{k\sigma, k'\sigma'}(t-t') = \delta_{\sigma\sigma'}\, \delta_{k,k'}\, \mathcal{G}(k, t-t')$$

$\hookrightarrow t \xleftarrow{k} t'$