$SU(2)$ algebra: α, etc: x, y, z

$$[S_i^\alpha, S_j^\beta] = i\,\delta_{ij}\,\varepsilon^{\alpha\beta\gamma} S_i^\gamma$$

tot. antisymm. symbol.

In terms of $S_i^\pm = S_i^x \pm i S_i^y$:

$$[S_i^z, S_j^\pm] = \pm\delta_{i,j} S_i^\pm \; ; \; [S_i^+, S_j^-] = 2\delta_{i,j} S_i^z$$

Rep. in terms of bosons: Holstein Primakov

For spin-S:

$$S_i^- = a_i^\dagger (2S - \hat{n}_i)^{\frac{1}{2}} \qquad S_i^+ = (2S - \hat{n}_i)^{\frac{1}{2}} a_i$$

$$S_i^z = S - \hat{n}_i \; ; \; \hat{n}_i = a_i^\dagger a_i \; ; \; [a_i, a_j^\dagger] = \delta_{ij}$$

Ex: show that $S_i^\pm, S_i^z$ satisfy $SU(2)$ alg!

A groundstate: $|\uparrow\uparrow\ldots\uparrow\rangle = |0,0,\ldots,0\rangle = |0\rangle$

has $S_i^z|0\rangle = S|0\rangle$, or $\hat{n}_i|0\rangle = 0$

Acting w/ $a_i^\dagger$, one finds $\Delta S_i^z \sim 1$, or $\frac{\Delta S_i^z}{S} \ll 1$

for large S: small fluctuations ~~in~~ around $|0\rangle$ are found by expanding in $1/S$.

Use HP on Heisenberg model (ex!).

$$H = -J\sum_i \underbrace{\tfrac{1}{2}\left(S_i^{+}S_{i+1}^{-} + S_i^{-}S_{i+1}^{+}\right)}_{= S_i^x S_{i+1}^x + S_i^y S_{i+1}^y} + S_i^z S_{i+1}^z$$

$$= -JS\sum_i \left(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i - 2a_i^\dagger a_i\right)$$

$$-NJS^2 + O(S^0)$$

Use FT: $a_j = \frac{1}{\sqrt{N}}\sum_k e^{-ikj} a_k$

$H = -JNS^2 + \sum_k \hbar\omega_k a_k^\dagger a_k + O(S^0)$, where

$\hbar\omega_k = 4JS\sin^2(k/2) \sim JSk^2$ for $k \to 0$.

So: low energy excitations have quadratic dispersion

$\rightsquigarrow$ magnons

Anti-ferromag. case: $H = +J\sum_i \vec{S}_i \vec{S}_{i+1}$ $J > 0$.

g.s. for bi-partite system: $\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow$

A B A B ...

To find bosonic rep, we first rotate spins on

B sublattice: $S_B^x \to \tilde{S}_B^x$; $S_B^y \to -\tilde{S}_B^y$; $S_B^z \to -\tilde{S}_B^z$,

So GS is like a ferromagnet again

The hamiltonian can be written as

$$H = -J\sum_j \tilde{S}^z_{2j-1}\tilde{S}^z_{2j} + \tilde{S}^z_{2j} S^z_{2j+1}$$
$$-\frac{1}{2}\left[S^+_{2j-1}\tilde{S}^+_{2j} + \tilde{S}^+_{2j} S^+_{2j+1} + S^-_{2j-1}\tilde{S}^-_{2j} + \tilde{S}^-_{2j} S^-_{2j+1}\right]$$

The $\tilde{S}^z_i$ are just spin operators, so we can drop the $\sim$, and perform a HP transformation as in the ferromagnetic case, which gives (in limit of large S)

$$H = -NJS^2 + JS\sum_i \left[a^\dagger_i a_i + a^\dagger_{i+1} a_{i+1} + a_i a_{i+1} + a^\dagger_i a^\dagger_{i+1}\right] + O(S^0)$$

We need to take care of the anomalous $a_i a_{i+1}$ terms.
First, one does a FT:

$$H = -NJS(S+1) + JS\sum_k (a^\dagger_k, a_{-k}) \begin{pmatrix} 1 & \cos k \\ \cos k & 1 \end{pmatrix} \begin{pmatrix} a_k \\ a^\dagger_{-k} \end{pmatrix} + O(S^0).$$

This can be diagonalised ~~with~~ with a Bogoliubov transformation, which gives $H = JS\sum_k \sqrt{1-\cos^2 k}\,\left[\alpha^\dagger_k \alpha_k + \frac{1}{2}\right]$,

so the spectrum is linear for small k: $\sqrt{1-\cos^2 k} = |\sin k| \to k$ for k small.

This result is true even ~~to~~ for small half integer spins!
Integer ~~a~~ case: system has a gap! No low energy ~~excitation~~ excitations.
Difference between integer & half integer case: Topological origin! Haldane

Jordan-Wigner:

Correspondence between spin-$\frac{1}{2}$ and a fermion:

$$|\downarrow\rangle = |0\rangle \; ; \quad |\uparrow\rangle = |1\rangle = f^{+}|0\rangle$$

So, $f^{+} = \sigma^{+}$; $f = \sigma^{-}$ $\rightsquigarrow$ $\sigma^{x} = \sigma^{+} + \sigma^{-}$

$\sigma^{z} = (2f^{+}f - 1)$ $\qquad \sigma^{y} = -i(\sigma^{+} - \sigma^{-})$

$\uparrow$
Pauli

Using commutators of Pauli-σ's, one finds that

$$\{f, f^{+}\} = 1 \; ; \quad \{f, f\} = \{f^{+}, f^{+}\} = 0$$

If we have more than one site,

σ's on different sites commute, f's anti commute!

This can be solved via 'string-operators'.

$$f_i = \underbrace{\Big(\prod_{j<i}(-\sigma_j^{z})\Big)}_{(-1)^{\sum_{j<i} n_j}} \sigma_i^{-} \; ; \qquad f_i^{+} = \Big(\prod_{j<i}(-\sigma_j^{z})\Big)\sigma_i^{+}$$

$=$ Parity # of fermions on sites $1, 2, \ldots, i-1$

These f_i's satisfy $\{f_i, f_j^{+}\} = \delta_{i,j}$;

$$\{f_i, f_j\} = \{f_i^{+}, f_j^{+}\} = 0$$

For ~~h~~ $J = h = 1$:

$$\varepsilon_k = \cancel{2}\sqrt{2-2\cos k} = 2\left|\sin \tfrac{k}{2}\right|,$$ so

we have a gap-closing at $k=0$, w/ linearly dispersing mode!

Notes: JW is a non-local transformation, so local properties in spin-language are non-local in fermion language. Spectrum of models is the same, but ~~fe~~ physics is different.
Starting from fermion model: topological SC, in spin language: ferromagnet!.. (Broken symmetry!).

Often, the effect of interactions (say ~~h=J small h~~, h-term if $h \ll J$, or J if $J \ll h$) has to be taken into account perturbatively:
We'll look into 'Green's functions, which will play an important role, in PT,: give physical observables!

J-W can be used to solve the 'Transverse-field Ising model'.

$$H = -J\sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x - h\sum_j \sigma_j^z \quad , \text{use } \sigma_{i+N}^\alpha = \sigma_i^\alpha$$

Phases: $h=0$; $J=1$ GS: $|\rightarrow\rightarrow\rightarrow\cdots\rightarrow\rangle$ and $|\leftarrow\leftarrow\leftarrow\cdots\leftarrow\rangle$.

Finite energy to create an excitation

$h=1$; $J=0$ GS: $|\uparrow\uparrow\cdots\uparrow\rangle$; not degenerate.

There should be a phase transition!

This occurs at $h=J$

To solve the model: use JW: gives fermionic model:

$$H = -h\sum_{j=1}^{N} (2f_i^\dagger f_i - 1)$$
$$+J\sum_{j=1}^{N} (f_i^\dagger f_{i+1}^\dagger + f_i^\dagger f_{i+1} + h.c.) \quad \text{OBS: boundary term.}$$

* Use fourier transform & Bogoliubov transformation:

$$H = \sum_k \varepsilon_k (2c_k^\dagger c_k - 1) \qquad \varepsilon_k = (J^2 + h^2 - 2Jh\cos k)^{1/2}$$