Quantum field theory for condensed matter

Introduction

Relativistic QFT:

Focus on particle/high energy physics

Calculation of: * scattering amplitudes

* decay rates

⋮

Involves 'a few' particles.

Systems in condensed matter & (modern) atomic physics: 'non-relativistic many body quantum mechanics', formulated as 'non relativistic QFT'.

Starting point: large number ($\sim 10^{20}$) of particles, whose microscopics is known (interacting electrons!)

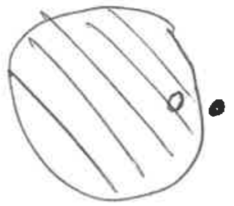
Task: find & solve the effective theory that describes the low energy behavior: often quite different from microscopics 'more is different'. ("emergent behavior").

Superfluids, superconductors, quantum Hall effect: understood ~~as~~ in terms of new, emergent 'order parameter fields', effective gauge fields, etc.

These states can be thought of as new 'vacuum' states, with their own set of 'particles' or excitations.

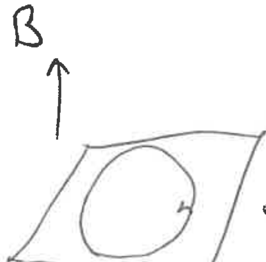
The dimension of the system plays an important role! (Why can CM systems really be 1+1 or 2+1 dimensional?)

* The excitations can be fermionic quasielectrons, holes: $q = \pm e$; gapless



* Quantised waves: sound: phonons, spin: magnon } Bosons, gapped or gapless

* 2d ~~e~~ electron gas:



  * chiral edge modes (gapless)
  * quantise vortices: fractional charge & statistics

  anyons

* 1d: notion of statistics not really defined

'Second quantisation' (misnomer, as we'll see).

Important postulate: existence of 'identicle particles':
bosons ~~and~~ or fermions w/ the same quantum numbers ((charge) (spin))
have sym. ~~&~~ anti-sym. wave functions: $\psi_B(x_1,x_2) = \psi_B(x_2,x_1)$
$\psi_F(x_1,x_2) = -\psi_F(x_2,x_1)$

For states this implies:

$$|\lambda_1,\lambda_2\rangle_{B,F} = \frac{1}{\sqrt{2}}\left(|\lambda_1\rangle_{\underset{\text{particle}}{1}}\otimes|\lambda_2\rangle_2 + J\,|\lambda_2\rangle_1\otimes|\lambda_1\rangle_2\right)$$

$$J = \begin{cases} +1 & B(\text{oson}) \\ -1 & F(\text{ermion}) \end{cases}$$

λ_i: label of states;

In general: $$|\lambda_1,\dots,\lambda_N\rangle = \frac{1}{\sqrt{N!\prod_i (n_{\lambda_i})!}}\sum_P \begin{Bmatrix} +1 & B \\ -1 & F \end{Bmatrix}^{\mathrm{sgn}P} |\lambda_{P_1}\rangle\otimes\dots\otimes|\lambda_{P_N}\rangle$$

P: sum over all permutations
sgn P: sign of the permutation, i.e. # pair swaps mod 2

These states live in a N particle Hilbert space: $\mathcal{F}^N$

Fermion case: Slater determinants $\left(\begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \varphi_3(x_2) \\ \varphi_1(x_3) & \varphi_2(x_3) & \varphi_3(x_3) \end{vmatrix} \text{ etc.}\right)$

Boson case: symmetrised monomials

Problem w/ Slaterdeterminants: cumbersome, member of particles can't fluctuate!

(Occupation) number representation:

So far: state labelled by q.n. of each particle:

$| 1 1 1 2 3 3 3 5 5 6 \rangle$

In terms of occupation numbers: $| 3 1 3 0 2 1 \ldots \rangle$

$\# \lambda_1$ $\# \lambda_2$

n_{λ_1}

entries is reduced.

Fermions: $n_{\lambda_i} = 0, 1$

General state in $\mathcal{F}^N$: $|\psi\rangle = \sum_{\{n_i\}} c_{n_1, n_2, \ldots} | n_1, n_2, \ldots \rangle$

sum $\sum_i n_i = N$

We can now allow for states w/ contributions ~~from~~ w/ different particle numbers: drop constraint on N.

States live in 'Fock space': $\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{F}^N$

$\mathcal{F}^0$: vacuum state w/ particles!

Creation operators: $a^\dagger : \mathcal{F}^N \to \mathcal{F}^{N+1}$

annihilation ": $a : \mathcal{F}^N \to \mathcal{F}^{N-1}$

Define:

$$a_i^+ |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i+1}\, \mathfrak{J}^{s_i} |n_1, \ldots n_i+1, \ldots\rangle$$

$s_i = \sum_{j=1}^{i-1} n_j$: takes care of signs (antisymmetrisation for fermions).

Also: for fermions n_i are def. modulo 2, so if $n_i = 1$, then $n_i + 1 = 0$

$$a_i |n_1, n_2, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\, \mathfrak{J}^{s_i} |n_1, n_2 \ldots n_i - 1, \ldots\rangle$$

Arbitrary states are build from vacuum: $|0\rangle = |0, 0, \ldots\rangle$

$$|n_1, n_2, \ldots\rangle = \prod_i \frac{(a_i^+)^{n_i}}{\sqrt{n_i!}} |0\rangle$$

In the case of bosons, we have: $[a_i, a_j] = [a_i^+, a_j^+] = 0$

$$[a_i, a_j^+] = \delta_{ij}$$

For fermions: $\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0$; $\{a_i, a_j^+\} = \delta_{ij}$

Show this!

F.I.: $(a_i^+ a_j^+ - \mathfrak{J} a_j^+ a_i^+) |n_1, n_2, \ldots\rangle = 0 \quad i \neq j$

$i = j$: $[a_i^+, a_i^+] = 0$ (bosons)

$(a_i^+)^2 = 0$ fermions