Quantum Field Theory for Condensed Matter - 2017
Exercise Set 3 (14 points)
Due date: friday, may 26st

1. For a system of electrons and phonons, the phonon part of the Hamiltonian is given by

$$H_{ph} = \sum_{\mathbf{q},j} \omega_{\mathbf{q}} a^\dagger_{\mathbf{q},j} a_{\mathbf{q},j} + \text{const.}$$

in which $a^\dagger_{\mathbf{q},j}$ is a bosonic creation operator for a phonon with momentum $\mathbf{q}$ and polarization along the j-axis and $\omega_{\mathbf{q}}$ is the energy of the phonon, which only depends on $|\mathbf{q}|$.
The electron-phonon interaction is given by the Hamiltonian

$$H_{el-ph} = \gamma \sum_{\mathbf{q}j} \frac{i\mathbf{q}.\hat{e}_j}{\sqrt{2m\omega_q}} n_{\mathbf{q}} \left(a_{\mathbf{q},j} + a^\dagger_{-\mathbf{q},j} \right)$$

Here m is the ion mass, γ is a coupling constant, $\hat{e}_j$ is the unit vector along the j-axis and $n_{\mathbf{q}} \equiv \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}}$ where c is the electron annihilation operator (for simplicity we ignore the spin of the electron.)

   a) (1 p) By setting the chemical potential for the phonons to zero and performing a Fourier transform to go to the frequency domain, formulate the coherent state action of the electron-phonon system. Take S_{el} as an electron action which you do not need to write out explicitly. Note that not only phonons are bosons, but $n_{\mathbf{q}}$ is a bosonic operator as well. Use the notation $n_q[\bar{\psi}, \psi] = \rho_q$ where $q = (\omega_n, \mathbf{q})$.

   b) (2 p) Integrate out the phonon fields, and show that the effective action for the electrons is given by

$$S_{eff}[\bar{\psi}, \psi] = S_{el}[\bar{\psi}, \psi] - \frac{\gamma^2}{2m} \sum_q \frac{|\mathbf{q}|^2}{\omega_n^2 + \omega_{\mathbf{q}}^2} \rho_q \rho_{-q}$$

2. In this exercise we calculate some features of the BCS gap equation,

$$\frac{1}{g\nu} = \int_0^{\omega_D} d\xi \frac{\tanh[\sqrt{\xi^2 + \Delta^2}/2T]}{\sqrt{\xi^2 + \Delta^2}},$$

in which g is the interaction strength, ν is the density of states at Fermi energy, ω_D is the Debye frequency, Δ is the gap and T is the temperature. Note that we set $k_B = 1$.

   a) (1 p) At $T = 0$, find the gap, Δ. Furthermore, for $g\nu \ll 1$, simplify your result.

   b) (1 p) At $T = T_c$ which is the critical temperature, the gap closes, i.e. $\Delta = 0$. Find the critical temperature. Hint: use the integration by parts and assume that $T_c \ll \omega_D$. You can use,

$$\int_0^\infty \frac{\log x}{\cosh^2 x} dx = \log \frac{\pi}{4} - \gamma,$$

   where $\gamma \simeq 0.577$ is Euler's constant.

   c) (2 p) For $T = T_c(1 - t)$ where $0 < t \ll 1$, the gap is small in comparison with the other energy scales. Expand the gap equation around T_c, and keep the lowest orders in t and Δ. Assume that $T_c \ll \omega_D$, and show that

$$\Delta = \kappa \sqrt{T_c^2 t} = \kappa \sqrt{T_c(T_c - T)}.$$

   Find the numerical value for κ as well.

3. In this exercise spin polarized $p_x + ip_y$ superconductor in $2D$ will be studied. The interaction term in this case reads,

$$V[\bar{\psi}, \psi] = \frac{g}{4} \int d^2 r \bar{\psi}(\nabla \bar{\psi}).\psi(\nabla \psi), \quad (1)$$

in which $\bar{\psi}(x, y)$ and $\psi(x, y)$ are Grassmann fields defined on the $2D$ plane and $\nabla = (\partial_x, \partial_y)$.

a) (1 p) Use the identity,

$$\int D[\bar{\mathbf{\Delta}},\mathbf{\Delta}]\exp\Big[-\frac{1}{g}\int d\tau d^2r\big(\bar{\mathbf{\Delta}}-\frac{g}{2}\bar{\psi}(\mathbf{\nabla}\bar{\psi})\big).\big(\mathbf{\Delta}+\frac{g}{2}\psi(\mathbf{\nabla}\psi)\big)\Big]\sim 1,$$

to replace the interaction term with an integral over the vector fields $\mathbf{\Delta}$ and $\bar{\mathbf{\Delta}}$ in the path integral. We assume that there is no external electromagnetic field, the chemical potential is μ and there is **no** *spin index* for the Grassmann fields. Write down the action, i.e. $S[\bar{\psi},\psi,\bar{\mathbf{\Delta}},\mathbf{\Delta}]$.

b) (1 p) Write the action in the following form,

$$S=\frac{1}{g}\int d\tau d^2r\bar{\mathbf{\Delta}}.\mathbf{\Delta}+\frac{1}{2}\int d\tau d^2r\begin{pmatrix}\bar{\psi} & \psi\end{pmatrix}\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}, \tag{2}$$

and determine the matrix M. It's elements contain derivatives.

c) (1 p) To find the saddle-point solution, assume that $\bar{\mathbf{\Delta}}$ and $\mathbf{\Delta}$ are constant vector fields on the plane. Perform the Fourier transformation as follows,

$$\psi(\mathbf{r},\tau)=\sum_q\psi(q)e^{i(\mathbf{q}.\mathbf{r}-\omega_n\tau)},\qquad \psi(q)=\frac{1}{\beta A}\int d\tau d^2r\psi(\mathbf{r},\tau)e^{-i(\mathbf{q}.\mathbf{r}-\omega_n\tau)}, \tag{3}$$

where A is the area of the plane, $q=(\omega_n,\mathbf{q})$ and note that $\bar{\psi}$ transforms like the "complex conjugate" of ψ. The action has the following form,

$$S=\frac{1}{g}A\beta\bar{\mathbf{\Delta}}.\mathbf{\Delta}+\frac{1}{2}A\beta\sum_q\begin{pmatrix}\bar{\psi}(q) & \psi(-q)\end{pmatrix}\begin{pmatrix}\tilde{M}_{11}(q) & \tilde{M}_{12}(q)\\ \tilde{M}_{21}(q) & \tilde{M}_{22}(q)\end{pmatrix}\begin{pmatrix}\psi(q)\\ \bar{\psi}(-q)\end{pmatrix}. \tag{4}$$

Determine the matrix $\tilde{M}(q)$.

d) (1 p) Do the functional integral over $\bar{\psi}$ and ψ and show that the effective action for $\bar{\mathbf{\Delta}}$ and $\mathbf{\Delta}$ is,

$$S_{eff}=\frac{1}{g}A\beta\bar{\mathbf{\Delta}}.\mathbf{\Delta}-\sum_q\log\big[\omega_n^2+\xi_{\mathbf{q}}^2+|\mathbf{\Delta}.\mathbf{q}|^2\big],$$

in which $\xi_{\mathbf{q}}=\frac{\mathbf{q}^2}{2m}-\mu$.

e) (1.5 p) Due to the symmetry of the model, one can always write $\mathbf{\Delta}=(\Delta_x,\Delta_y e^{i\theta})$, in which Δ_x, Δ_y and θ are real numbers. Write the saddle point equations for $\mathbf{\Delta}$ and argue that in order to have consistent equations, there are only two possible cases, $\theta=\pm\frac{\pi}{2}$, or equivalently $\mathbf{\Delta}=\Delta(1,\pm i)$.

f) (1 p) At $T=0$, use $\mathbf{\Delta}=\Delta(1,\pm i)$ to find Δ as a function of g, ω_D, p_F(the Fermi momentum) and $\nu(0)$ (the density of states at Fermi energy).

Hint: First do the Matsubara sum. To perform the integral over $\mathbf{q}$, assume that $\omega_D\ll\mu$.

g) (0.5 p) Based on the poles that appeared in the Matsubara sum, can you write the dispersion relation for the excitations, namely the Bogoliubov quasiparticles?