Quantum Field Theory for Condensed Matter - 2017
Exercise Set 2 (14 points)
Due date: monday, may 8th

1. (1 p) The Grassmann numbers ξ_1, ξ_2 and ξ_3 satisfy the usual anti-commutation relations, i.e. $\{\xi_i, \xi_j\} = 0$. Evaluate the following integrals

$$\int d\xi_1 d\xi_2 \sin(\xi_1 + \xi_2) e^{\xi_2}, \qquad \int d\xi_1 \cos(\xi_1 + \xi_2)\cos(\xi_1 + \xi_3).$$

2. The two-point correlation function in imaginary-time is defined as follows,

$$\mathcal{C}_{AB}(\tau, \tau') \equiv -\langle \mathcal{T}_\tau [A(\tau) B(\tau')] \rangle,$$

where

$$\mathcal{T}_\tau [A(\tau)B(\tau')] = \theta(\tau - \tau') A(\tau) B(\tau') + \zeta\, \theta(\tau' - \tau) B(\tau') A(\tau),$$

with $\zeta = +1$ for bosonic operators, and $\zeta = -1$ for fermionic operators. To ensure convergence of $\mathcal{C}_{AB}(\tau, \tau')$, we should have

$$-\beta < \tau - \tau' < \beta.$$

a) (1 p) Show that

$$\mathcal{C}_{AB}(\tau, \tau') = \mathcal{C}_{AB}(\tau - \tau', 0).$$

This means that the correlation function only depends on the time difference so one can assume that $\tau' = 0$.

b) (1 p) Defining

$$\mathcal{C}_{AB}(\tau) \equiv \mathcal{C}_{AB}(\tau, 0),$$

show that

$$\mathcal{C}_{AB}(\tau - \beta) = \zeta \mathcal{C}_{AB}(\tau), \qquad \tau > 0, \tag{1}$$

$$\mathcal{C}_{AB}(\tau + \beta) = \zeta \mathcal{C}_{AB}(\tau), \qquad \tau < 0. \tag{2}$$

Hint: prove one of these relations, and show that the other follows.

3. (1 p) Consider the following Hamiltonian,

$$H = \epsilon f^\dagger f,$$

where ϵ is a constant and f is a fermionic operator. The partition function is,

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}[e^{-\beta(H - \mu N)}] \\ &= \sum_n \langle n | e^{-\beta(H-\mu N)} | n \rangle, \end{aligned}$$

where β is the inverse temperature and μ is the chemical potential. What are the possible values for N or equivalently, the possible states $|n\rangle$? Sum over all possible values and find the partition function for single state.

4. Consider the general quartic Hamiltonian,

$$H(a^\dagger, a) = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l,$$

where the operators are either bosonic or fermionic. The partition function is,

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}[e^{-\beta(H-\mu N)}] \\ &= \int d(\bar{\psi}, \psi) e^{-\sum_i \bar{\psi}_i \psi_i} \langle \zeta \psi | e^{-\beta(H - \mu N)} | \psi \rangle. \end{aligned}$$

a) (1 p) Divide β to into $\mathcal{N}$ pieces, insert *identities* in between these and send $\mathcal{N} \to \infty$ to obtain path integral in the continuum limit,

$$\mathcal{Z} = \int D(\bar{\psi}, \psi) e^{-S[\bar{\psi},\psi]}, \qquad S[\bar{\psi}, \psi] = \int_0^\beta d\tau [\bar{\psi}\partial_\tau \psi + H(\bar{\psi}, \psi) - \mu N(\bar{\psi}, \psi)]. \tag{3}$$

Mention the boundary conditions.

b) (1 p) Note that $\bar{\psi}$ and ψ are *dimensionless* and we know the Grassmann algebra for *dimensionless* numbers. To keep this feature, perform the Fourier transform as follows,

$$\psi_i(\tau) = \sum_{\omega_n} \psi_{i,n} e^{-i\omega_n \tau}, \qquad \psi_{i,n} = \frac{1}{\beta} \int_0^\beta d\tau \psi_i(\tau) e^{i\omega_n \tau}, \tag{4}$$

and write the action, S, in terms of $\bar{\psi}_{i,n}$ and $\psi_{i,n}$.

c) (1 p) From now on set all quartic terms to zero, i.e. $V_{ijkl} = 0$. Furthermore assume that the matrix h_{ij} has eigenvalues ϵ_a (a labels the eigenvalues). Do the the functional integral, and find the free energy as a sum over a and the Matsubara frequencies.

d) (1 p) Using the imaginary plane trick, perform the Matsubara sum.

e) (0.5 p) From your result of Ex. 3), can you argue for your result in part d)?

5. Consider a fermi gas with Coulomb interaction, $V(\mathbf{r}) = \frac{e^2}{|\mathbf{r}|}$, in a 3D box of size L at temperature T.

a) (1 p) Justify the Random Phase Approximation (RPA) and mention when it is applicable. Write the free energy F_{RPA} as a function of $V(\mathbf{q})$ and the polarization operator, $\Pi(q)$,

$$\Pi(q) \equiv \frac{2T}{L^3} \sum_p G_p G_{p+q}.$$

Note that $p = (p^0, \mathbf{p})$ and $q = (q^0, \mathbf{q})$. You can start your calculation from Eq.5.26 of Altland and Simons, but explain its meaning and structure.

b) (1 p) Do the sum over Matsubara frequencies p^0 and show that,

$$\Pi(q) = \frac{2}{L^3} \sum_{\mathbf{p}} \frac{n_F(\xi_{\mathbf{p}+\mathbf{q}}) - n_F(\xi_{\mathbf{p}})}{-iq^0 + \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}}},$$

where $n_F(\epsilon)$ is Fermi-Dirac distribution.

c) (1.5 p)At $T = 0$, approximate all the expressions up to first order in $\frac{|\mathbf{q}|}{q^0}$, go to continuum limit and finally rotate back to real time, i.e. replace $iq^0 \to \omega + i0^+$ to show that

$$\Pi(\omega, \vec{q}) \simeq 2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{1}{\omega} + \frac{\mathbf{p} \cdot \mathbf{q}}{m\omega^2} \right) \frac{\mathbf{q} \cdot \mathbf{p}}{|\mathbf{p}_F|} \delta(|\mathbf{p}| - p_F). \tag{5}$$

Here we dropped $i0^+$.

d) (1.5 p) Do the integral of part c), and express $\Pi(\omega, \mathbf{q})$ as a function of ω, $\mathbf{q}$, m (the electron mass) and n (the density of electrons).

e) (0.5 p) One can show that an *effective* potential for electrons with these approximations is,

$$V_{eff}(\omega, \mathbf{q}) = \frac{V(\mathbf{q})}{1 - V(\mathbf{q})\Pi(\omega, \mathbf{q})}.$$

The denominator has a pole at $\omega = \omega_p$ (known as the **plasma frequency**) which is a signature of an instability. Determine ω_p.