Quantum Field Theory for Condensed Matter - 2017
Exercise Set 1 (14 points)
Due date: monday, april 10th

1. (0.5 p) Show that for a set of operators A, B and C

   a)
   $$[AB, C] = A[B, C] + [A, C]B. \tag{1}$$
   This can be useful for bosonic systems.

   b)
   $$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{2}$$
   This can be useful for fermionic systems.

2. (0.5 p) Fermion creation ($c^\dagger_\sigma(\mathbf{r})$) and annihilation ($c_\sigma(\mathbf{r})$) operators in real space obey the anti-commutation algebra,
   $$\{c_\sigma(\mathbf{r}), c^\dagger_{\sigma'}(\mathbf{r}')\} = \delta_{\sigma\sigma'}\delta(\mathbf{r} - \mathbf{r}'), \qquad \{c_\sigma(\mathbf{r}), c_{\sigma'}(\mathbf{r}')\} = 0. \tag{3}$$
   Assume that the operator $a^\dagger_{i\sigma}$ creates an electron in the state $|W_i\rangle$ which is a Wannier state localized at site i of the lattice. Using a change of basis and the orthonormality of the Wannier states, show that these operators obey the fermionic algebra as well,
   $$\{a_{i\sigma}, a^\dagger_{j\sigma'}\} = \delta_{\sigma\sigma'}\delta_{ij}, \qquad \{a_{i\sigma}, a_{j\sigma'}\} = 0. \tag{4}$$

3. a) (0.5 p) Consider a general quadratic Hamiltonian
   $$H = \sum_{ij} a^\dagger_i \mathcal{H}_{ij} a_j, \tag{5}$$
   where $a^\dagger_i$ could be a boson or fermion creation operator. What is the constraint on $\mathcal{H}$ such that the hamiltonian H is hermitian?

   b) (0.5 p) Assume that a_i is either a boson or fermion annihilation operator. If we perform a linear transformation like,
   $$d_i = \sum_j M_{ij} a_j, \tag{6}$$
   what requirement do we have to put on M in order to preserve the algebra, i.e. the operators d_i obey the same bosonic or fermionic algebra.
   Furthermore, show that the discrete Fourier transform,
   $$a_{k\sigma} = \frac{1}{\sqrt{\mathcal{N}}} \sum_i e^{-i\mathbf{k}.\mathbf{r}_i} a_{i\sigma}, \tag{7}$$
   where $\mathcal{N}$ is the number sites and $\mathbf{r}_i$ is the position of the ith lattice point, meets the requirement.

4. a) (1.5 p) Take a square lattice of size $N \times N$ with lattice spacing a and set $\mathcal{H}_{ij} = -t$ for nearest neighbours and zero otherwise in the Exercise 3.a) for a fermionic system. Using a discrete Fourier transformation calculate eigenvalues of the Hamiltonian $\epsilon_{\mathbf{k}}$.

   b) (0.5 p) Sketch the contours of constant $\epsilon_{\mathbf{k}}$ in the Brillouin zone and note the geometry at half-filling (i.e., the Fermi surface obtained by filling exactly half of all available states).

5. (2.0 p) Consider the translational invariant two-body interaction,

$$\hat{V} = \frac{1}{2}\int d^dx d^dy \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})V(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}). \tag{8}$$

Perform a Fourier transformation and rewrite the two-body interaction $\hat{V}$ in momentum space in terms of

$$\tilde{V}(\mathbf{q}) = \int d^dr e^{-i\mathbf{q}.\mathbf{r}}V(\mathbf{r}). \tag{9}$$

6. Consider a spin chain in which at each site there is a spin-S 'particle'.

a) (0.5 p) Recall that the $SU(2)$ algebra is:

$$[S_m^+, S_n^-] = 2\delta_{mn}S_m^z, \qquad [S_m^z, S_n^\pm] = \pm\delta_{mn}S_m^\pm, \tag{10}$$

where $S_m^\pm = S_m^x \pm iS_m^y$. Show that this algebra can be represented by the following expressions known as the **Holstein-Primakoff** transformation:

$$S_m^+ = (2S - a_m^\dagger a_m)^{\frac{1}{2}} a_m, \qquad S_m^- = a_m^\dagger(2S - a_m^\dagger a_m)^{\frac{1}{2}}, \qquad S_m^z = S - a_m^\dagger a_m, \tag{11}$$

in which a_m is a bosonic annihilation operator at site m, obeying the bosonic algebra.

b) (0.5 p) Using part a, elaborate on the connection between the "harmonic oscillator" states constructed by $a^\dagger$ and the spin state at each site. Can this harmonic oscillator accept any occupation number? What are the translations of the states $S_z = -S$ and $S_z = +S$ to the harmonic states?

c) (1.0 p) Apply the Holstein-Primakoff transformation and a Taylor expansion for large S to the Ferromagnetic Heisenberg hamiltonian and derive the dispersion relation for spin waves in this case.

7. Consider a general linear transformation, a so called Bogoliubov transformation,

$$d_i = \sum_j A_{ij}c_j + B_{ij}c_j^\dagger.$$

a) (1.0 p) Show that the requirements for d_i to have the same bosonic or fermionic algebra as c_i, *i.e.*,

$$\left[d_i, d_j^\dagger\right]_\pm = \delta_{ij}, \qquad [d_i, d_j]_\pm = 0,$$

where $[\,,\,]_-$ denotes a commutator and $[\,,\,]_+$ denotes an anti-commutator, are the following ones:

$$AA^\dagger \pm BB^\dagger = 1 \qquad AB^T \pm BA^T = 0,$$

where $+$ is fermions and $-$ for bosons.

b) (1.0 p) For the fermionic case, show that AB^T is antisymmetric and that the first requirement in part a) can be satisfied by setting

$$A_{ij} = \cos\theta_i U_{ij}, \qquad B_{ij} = \sin\theta_i V_{ij}$$

where U and V are unitary matrices and the θ_i are real.

c) (1.0 p) For the bosonic case, show that AB^T is symmetric and that the the first requirement in part a) can be satisfied by setting

$$A_{ij} = \cosh\theta_i U_{ij}, \qquad B_{ij} = \sinh\theta_i V_{ij}$$

where U and V are unitary matrices and the θ_i are real.

8. Take the general quadratic Hamiltonian for spinless fermions involving two sites,

$$H = \sum_{ij=1,2} c_i^\dagger h_{ij} c_j + \Delta(c_1c_2 + c_2^\dagger c_1^\dagger), \tag{12}$$

where h_{ij} is a matrix obeying the constraint that you found in Exercise 3. a) and Δ is a real constant.

a) (1.0 p) Using the notation of Exercise. 7, consider the following Bogoliubov transformation:

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 \\ 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 & \cos\theta \\ \cos\theta & 0 \end{pmatrix} \begin{pmatrix} c_1^\dagger \\ c_2^\dagger \end{pmatrix} \tag{13}$$

and show that it satisfies the requirements of Exercise 7. a) for fermions for real θ. In part c) we will determine θ.

b) (1.0 p) Invert the transformation and write down c_1 and c_2 in terms of d_1,$d_1^\dagger$, d_2, $d_2^\dagger$ and θ.

c) (1.0 p) Insert what you found in part b) into the hamiltonian, and find θ such that coupling in front of terms like $d_1 d_2$ and $d_1^\dagger d_2^\dagger$ vanish. This means that the hamiltonian can be written as,

$$H = \sum_{ij=1,2} d_i^\dagger \tilde{h}_{ij} d_j. \tag{14}$$