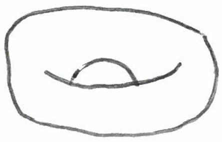
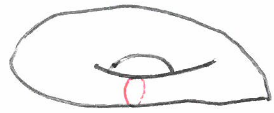
One the plane, the product $\prod_{\text{plaq } p} (\mathbb{1}+B_p)$

generates all possible loop configurations, which are all independent.

$\Rightarrow$ One, unique ground state.

On a torus, one has: $\prod_{\text{all plaq } p} B_p = \mathbb{1}$.

So, there are ~~4~~ loop configuration one can <u>not</u> obtain from $|\uparrow\uparrow\cdots\uparrow\rangle$.

$|\uparrow\cdots\uparrow\rangle =$  ;  ; 

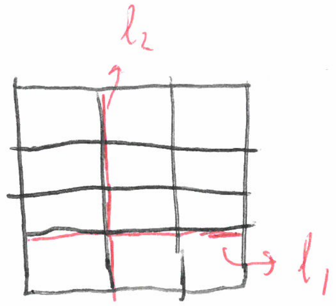
Take one of these; the others <u>can not</u> be ~~of~~ obtained by ~~applying~~ operating w/ a product of B_p's!

~~We~~ ~~X~~ On the torus, there are <u>four</u> ground states, not just one!

To create the loops, we define so-called Wilson operators:

$$W(s) = \prod_{j \in l_s} \sigma_j^x$$

$(s=1,2)$

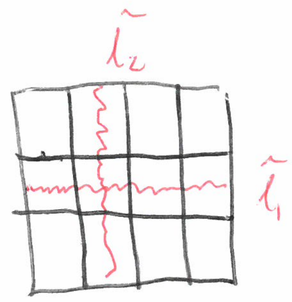


l_1, l_2: non-contractible loops around handle of the torus.

We have $[H_{TC}, W(s)] = 0$; $[W(1), W(2)] = 0$.

Eigenvalues of $W(s)$: ± 1.

We can also define $\hat{W}(s) = \prod_{j \in \hat{l}_s} \sigma_j^z$



$\hat{l}_s$: path on the dual lattice

We also have: $[H_{TC}, \hat{W}(s)] = [\hat{W}(1), \hat{W}(2)] = 0$,

but $W(1)\hat{W}(2) = -\hat{W}(2)W(1)$ and

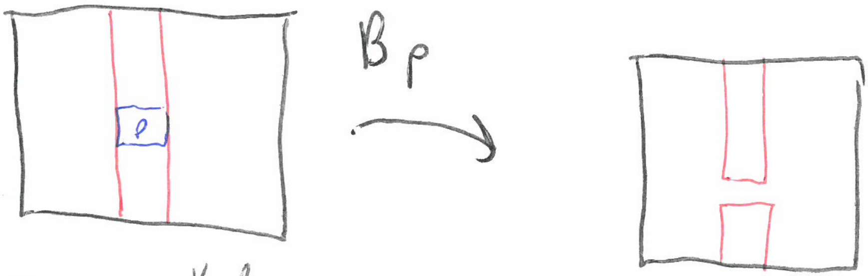
$W(2)\hat{W}(1) = -\hat{W}(1)W(2)$

What do they do?

$W(1), W(2)$: create a single, non-contractible loop around handle of torus.

$\hat{W}(1), \hat{W}(2)$: measure # of non-contractible loops, modulo 2!

A product of B_p's can only change the number of non-cont. loops by an even number!



non-cont. loops $=2$

non-cont. loops $=0$.

The number of G.S. on the torus are the ~~poss~~ # of different eigenvalues of $\hat{W}(1)$ and $\hat{W}(2)$,

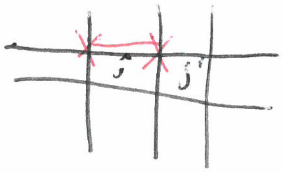
So 4 G.S.'s.

To create one from another, we act with $W(1)$, or $W(2)$.

There is a close connection w/ the torus degeneracy and the excitations of the model!

Excited states: have $A_s=-1$ or $B_p=-1$ for some star or plaquette.

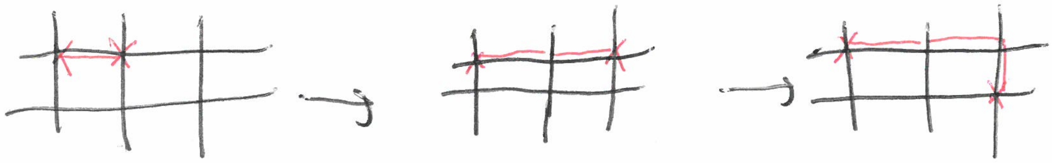
To create an excitation of the star operator: act with $\sigma_j^x |0\rangle$:



Two stars adjacent to j have e.v. (-1), so the energy is $2(2J_A) = 4J_A$, each excitation costs $2J_A$.

As

By acting with $\sigma_{j'}^x$, we can move an excitation, w/o. energy cost:



etc.

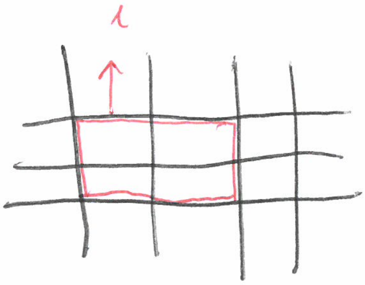
$\Delta E = 4J_a$ $\qquad$ $\Delta E = 4J_a$ $\qquad$ $\Delta E = 4J_A$

We def. 'electric' path operator: $w_l^{(e)} = \prod_{j\in l} \sigma_j^x$, with star excitations at the end of the string!

If ~~to~~ l is a closed loop: $w_l^{(e)}|0\rangle$ is again a ground state.

We have $w_l^{(e)}|0\rangle = |0\rangle$, if l is a contractible loop (not around handles of a torus!)

$w_l^{(e)}|0\rangle =$



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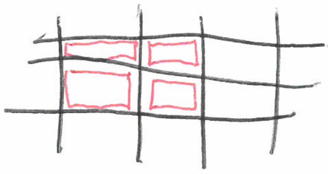
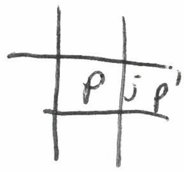


$= \prod_{p\in l} B_p|0\rangle = +|0\rangle$

If l winds around handles of the torus, $W_l^{(e)}|0\rangle$ is a different gs, not equal to $|0\rangle$.

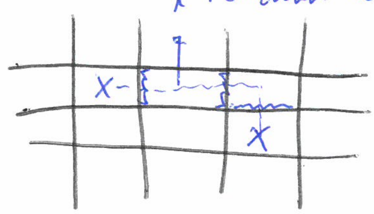
Excitation of plaquette operator B_p:
act w/ $\sigma_j^z|0\rangle$:



$$B_p\sigma_j^z|0\rangle \underset{j\in p}{=} -\sigma_j^z B_p|0\rangle$$
$$= -\sigma_j^z|0\rangle,$$ so B_p (or $B_{p'}$) have e.v. -1 after acting w/ σ_j^z.

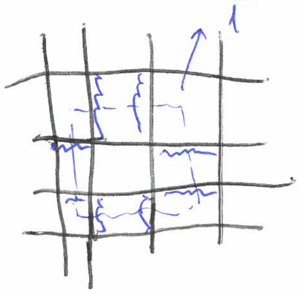
ΔE is now: $\Delta E = 4J_B$, and we can move 'magnetic' excitations around, w/ a magnetic string operator:

$$W_{\hat l}^{(m)} = \prod_{j\in\hat l}\sigma_j^z \; :$$



Again, if $\hat l$ is a closed loop, $W_{\hat l}^{(m)}|0\rangle$ is a ground state, equal to $|0\rangle$ if $\hat l$ is contractible, otherwise a different gS:

$W_{\hat l}^{(m)}|0\rangle =$



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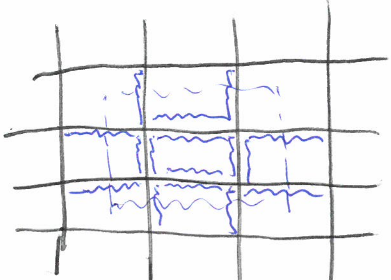


$$= \prod_{s\in\hat l} A_s|0\rangle$$
$$= |0\rangle$$

4-fold ~~gs~~ deg on torus:

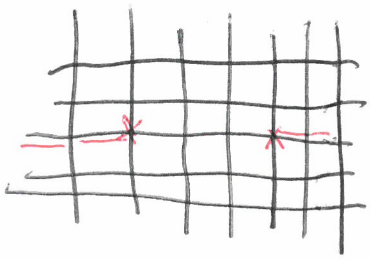
Take $|0\rangle$, w/o loops.

Create pair of star excitation, move one around torus annihilate, gives different g.s.

One can do the same w/ plaquette, or comb. of star & plaquette. This gives 4 ground states!

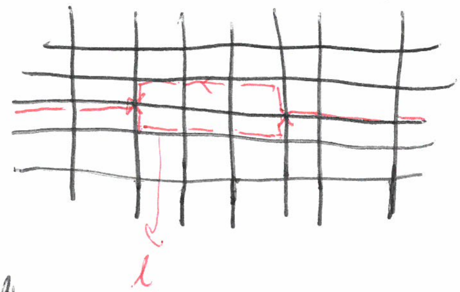
Statistics of the excitations:

Create a state w/ 2 e's: $|\chi_e\rangle$



$= |\chi_{2e}\rangle$

Exchange them, via $\hat{l}$:



After exchange:

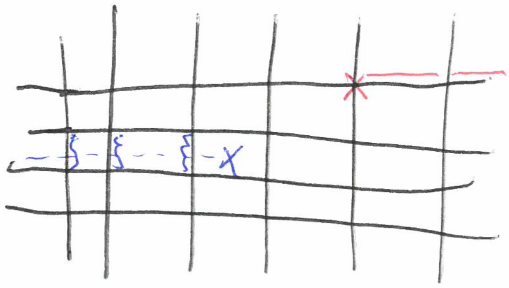
$$\prod_{j\in l} \sigma_j^x |\chi_{2e}\rangle = \prod_{p\in l} B_p |\chi_{2e}\rangle$$
$$= |\chi_{2e}\rangle.$$

The e particles are bosons!

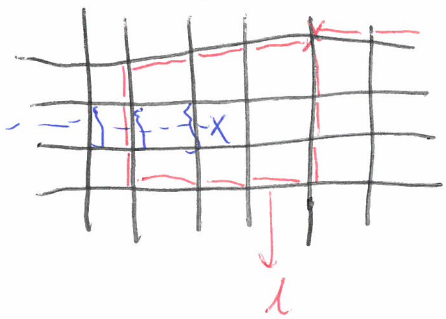
A similar argument gives that two m's are bosons!

Mutual statistics: lets move an e completely around an m!

State before braid:

$|\chi_{em}\rangle =$ 

State after braiding:



$$= \prod_{j\in l} \sigma_j^x |\chi_{em}\rangle$$

$$= \prod_{p\in l} B_p |\chi_{em}\rangle$$

$= -|\chi_{em}\rangle$, one of the plaquettes (w/ $\boxed{x}$) has $B_p|\chi_{em}\rangle = -|\chi_{em}\rangle$.

So, taking an e completely around an m gives a minus sign! Non-trivial mutual statistics!

We can also consider a 'bound e,m pair'. Lets call this ε; If we exchange two ε's, we get a minus sign! So, it is a fermion. Combination of two bosons gives a fermion!

Toric code is a model of spins.

* Does not have spin ordering:

  $\langle \sigma_i^z \sigma_j^z \rangle$ ~~ett~~ etc. is zero in GS.

* 4-fold deg. on torus.

* 4 types of excitations : * no exc. : 1
  * electric
  * mag.
  * (e, m) pair ε : fermion

Two e's at same place ~~with~~ annihilate :

$$e \times e = 1 \qquad e \times \varepsilon = m$$
$$m \times m = 1 \qquad m \times \varepsilon = e$$
$$e \times m = \varepsilon \qquad \varepsilon \times \varepsilon = 1$$