Generic phase diagram:

System is trivial if $|\mu| > 2|t|$

System is top. SC. if $|\mu| < 2|t|$, and $|\Delta| \neq 0$

($\Delta = 0$: ~~od~~ metallic)

If $|\mu| < 2|t|$, the Majorana bound state are exp. localized at left & right edge of the system. (not only on 1st & last site).

For periodic boundary conditions (L sites) we do a FT:

$$a_j = \frac{1}{\sqrt{L}} \sum_k c_k e^{ijak} \qquad a = \frac{2\pi}{L}$$

$j = 0, 1, \ldots L-1$

$k = 0, 1, \ldots L-1$

This gives: ($\Delta \in \mathbb{R}$)

$$H = -\frac{\mu}{2} \sum_k (c_k^+ c_k - c_{-k} c_{-k}^+)$$
$$- t \sum_k \cos(ak) \left[c_k^+ c_k - c_{-k} c_{-k}^+\right]$$
$$+ i\Delta \sum_k \sin(ak) \left[c_{-k} c_k - c_k^+ c_{-k}^+\right]$$

In terms of $\Psi_k^+ = (c_k^+, c_{-k})$ $\Psi_k = \begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix}$, we have

$$H = \sum_k \Psi_k^+ \mathcal{H}_k \Psi_k \qquad \mathcal{H}_k = (-\mu/2 - t\cos(ak))\,\tau_z$$
$$+ \Delta \sin(ak)\,\tau_y$$

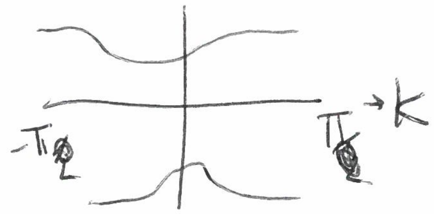
τ_z: Pauli matrices in spinor-space (particle-hole).

Eigenvalues of ε of $d_x \tau_x + d_y \tau_y + d_z \tau_z$:

$$\varepsilon^2 = d_x^2 + d_y^2 + d_z^2,$$

So we have $\varepsilon_k^2 = \left(-\frac{\mu}{2} - t\cos(ak)\right)^2 + \Delta^2 \sin^2(ak)$

Bands:



$K = \frac{2\pi k}{L}$

Transition between trivial and top. phase:

$\varepsilon_k = 0$ for some k.

So, we need $\sin^2\left(\frac{2\pi k}{L}\right) = 0 \wedge \left(-\frac{\mu}{2} - t\cos\left(\frac{2\pi k}{L}\right)\right)^2 = 0$

$k = 0$ or $k = L/2$

$k = 0$ gives $-\frac{\mu}{2} - t\cos(0) = 0 \rightsquigarrow \mu = -2t$

$k = \frac{L}{2}$: $-\frac{\mu}{2} + t = 0$ $\quad \mu = +2t$

Top. phase: $|\mu| < 2|t|$, trivial phase: $|\mu| > 2|t|$
($\Delta \neq 0$)

How can we 'detect' (theoretically) if the system is topological?

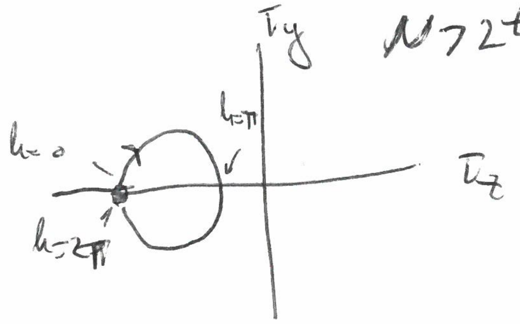
We write $\mathcal{H}_k = \vec{d}_k \cdot \vec{\tau}$
↳ vector of Pauli matrices

$\vec{d}_k = \left(0, \Delta \sin(ak), \left(-\frac{\mu}{2} - t\cos(ak)\right)\right)$.

We can plot $\vec{d}_k$ in $\tau_y \tau_z$ plane.

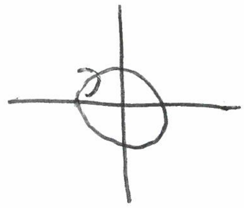
Third phase: $\mu, t, \Delta > 0$

$\mu > 2t$

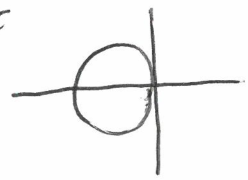


Curve does <u>not</u> wind around the origin

$\mu < 2t$: Curve does wind



$\mu = 2t$



Gap closes, so curve through origin, winding <u>not</u> defined

The winding of the 2-component $\vec{d}$ vector is a 'topological invariant', taking values in $\mathbb{Z}$ in general.

[More mathematically:

$$\hat{d}_k = \frac{\vec{d}_k}{|\vec{d}_k|}$$: map from S_1 (BZ) to S_1 : these maps are characterised by an integer, the winding number.

Phases w/ different winding number are top. distinct.

Note: this requires that $\vec{d}$ has a zero component!

(This is part of a much more general classification scheme)]

'Explanation' of the Majorana bound states:

In the top. phase: winding number $= \pm 1$

Outside of the system: there is just a trivial vacuum; winding is zero.

A change in winding number can only occur if the gap closes, so, at the boundary, the gap must close. ~~This~~

In this system, this ~~is~~ leads to a Majorana Bound state.

If one had next nearest neighbour hopping and pairing only: winding# is ± 2, so 2 Majorana Bound states at the edge.

Note: this behavior changes if Δ is complex and varies.

Toric code model:

Solvable model of interacting spins.

~~Sh It has a to~~

It describes a topological phase, with 'anyonic' excitations.

Spin-$\frac{1}{2}$: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We work in σ^z basis.

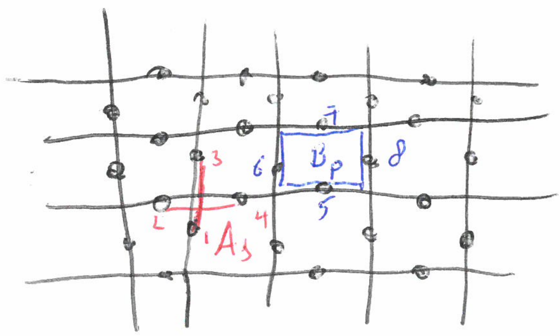
Spins on different sites commute: $[\sigma_i^\alpha, \sigma_j^\beta] = 0 \quad i \neq j$

$\alpha, \beta = x, y, z$

On same site: $(\sigma_i^\alpha)^2 = 1$, $\{\sigma_i^\alpha, \sigma_i^\beta\} = 0 \quad \alpha \neq \beta$

$\sigma_i^x \sigma_i^z = -\sigma_i^z \sigma_i^x$ etc.

Spins live on edges of square lattice (open for now):



The operators in H:

$$A_s = \prod_{j \in s} \sigma_j^z = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$$

$$B_p = \prod_{j \in p} \sigma_j^x = \sigma_5^x \sigma_6^x \sigma_7^x \sigma_8^x$$

A_s: star operator; B_p: plaquette operator

↳ 4 spins around the vertex

4 spins around the plaquette.

The toric code Hamiltonian:

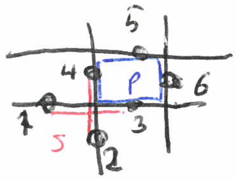
$$H_{TC} = -J_A \sum_{\substack{\text{all } s \\ \text{stars}}} A_s - J_p \sum_{\substack{\text{all} \\ \text{plaq. } p}} B_p$$

<u>All</u> operators A_s, B_p commute:

$$[A_s, A_{s'}] = [B_p, B_{p'}] = 0$$ (σ^z's commute, σ^x's commute).

$$[A_s, B_p] = 0$$ if s and p are <u>not</u> adjacent (<u>no</u> common spins).

Non-trivial case:



$$A_s B_p = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z \sigma_3^x \sigma_4^x \sigma_5^x \sigma_6^x$$

$$= (-1)^2 \sigma_3^x \sigma_4^x \sigma_5^x \sigma_6^x \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z = B_p A_s$$

two common spins, commuting gives $2 \times (-1)$ factor.

All operators commute, so they can be diagonalised ~~at~~ simultaneously!

The eigenvalues: $A_s^2 = +1 \Rightarrow$ ev. are ± 1

$B_p^2 = +1 \quad \Uparrow$

With J_A, J_B positive, the groundstate should have:

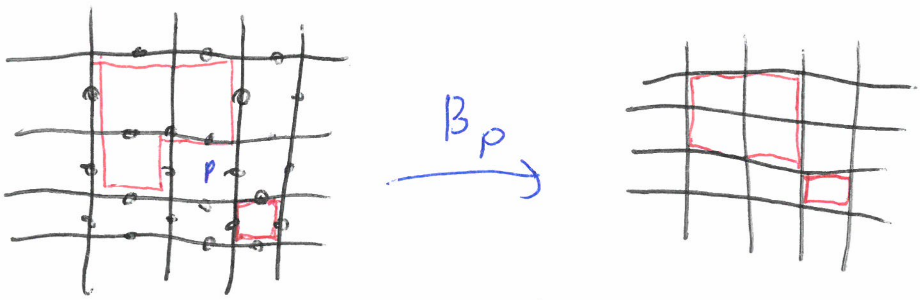
$$A_s|0\rangle = |0\rangle \, ; \; B_p|0\rangle = |0\rangle$$

Lets start w/ A_s:

It has eigenvalue +1 if 4 spins are up, 2 up & ~~4~~ 2 down, 0 up, 4 down, or:

even ~~#~~number of downs

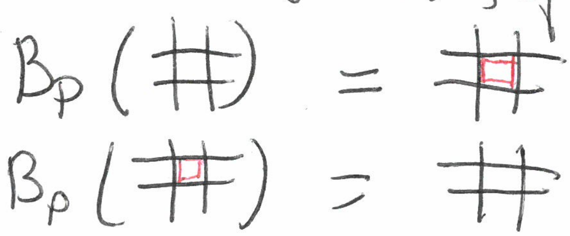
All conf. w/ even number of ↓ spins around each vertex are a ground state of $-J_A \sum_s A_s$.
If we colour the down states, we get closed loops; ex:

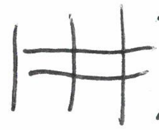
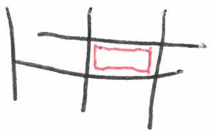


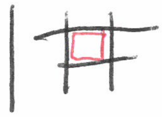
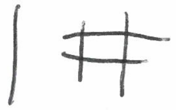
This is **not** a g.s. of $-J_B \sum_p B_p$

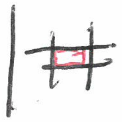
B_p: flips all the spins around a plaquette (σ^x operators).

What are eigenstates of B_p (w/ ev +1)?



So: $B_p \Big(\Big|$  $\Big) + \Big|$  $\Big) \Big)$

$= \Big|$  $\Big) + \Big|$  $\Big)$

$\Big|$  $\Big) \pm \Big|$  $\Big)$ eigenstate of B_p w/ eigenvalue ± 1

The g.s. of H_{TC} (on plane) is the equal-amplitude superposition of all 'loop configurations':

Acting w/ B_p: does not change the eigenvalues of the A_s.

So, look at reference state ~~to~~ without loops, i.e. all spins up: $|\uparrow\uparrow\cdots\uparrow)$.

Take the state: $\prod_p (\mathbb{1} + B_p)\,|\uparrow\cdots\uparrow)$

This is an eigenstate of all A_s's.

Lets act w/ a $B_{p'}$, we need to show that it has eigenvalue $+1$:

$$B_{p'} \prod_p (\mathbb{1} + B_p)\,|\uparrow\cdots\uparrow) = \prod_{p\neq p'} (\mathbb{1} + B_p)\, B_{p'} (\mathbb{1} + B_{p'})\,|\uparrow\cdots\uparrow)$$

$$= \prod_{p\neq p'} (\mathbb{1} + B_p)\,(B_{p'} + \mathbb{1})\,|\uparrow\cdots\uparrow) = \prod_p (\mathbb{1} + B_p)\,|\uparrow\cdots\uparrow)$$

$(B_{p'}^2 = \mathbb{1})$