So, we approximate the field ψ_0 by a constant classical value. $N_0 = \langle a_0^\dagger a_0 \rangle$ is macroscopically large, but $[a_0, a_0^\dagger] = 1$, so the fluctuations are small! (so we can neglect the commutator).

Adding weak interactions:

The addition of weak interactions (adiabatically) does not change the fact that ψ_0 is macroscopically occupied. So it dominates the action:

$$T\, S[\bar{\psi}_0, \psi_0] = -\mu \bar{\psi}_0 \psi_0 + \frac{g}{2L^d}(\bar{\psi}_0 \psi_0)^2, \quad w/\ g > 0.$$

Because of the interaction, we can interpret ψ_0 as a field again, and μ can be larger than zero.

We can now do a saddlepoint approximation.

$$\frac{\delta}{\delta \psi_0} S[\bar{\psi}_0, \psi_0] = 0 = -\frac{\mu}{T}\bar{\psi}_0 + \frac{g}{TL^d}\bar{\psi}_0 \bar{\psi}_0 \psi_0, \text{ or}$$

$$\bar{\psi}_0 \left(-\mu + \frac{g}{L^d}\bar{\psi}_0 \psi_0\right) = 0$$

* For $\mu < 0$, we need $\bar{\psi}_0 = 0 \to \psi_0 = 0$, <u>no</u> condensate.

For $\mu > 0$: $\bar{\psi}_0 \neq 0$, and $|\psi_0| = \sqrt{\frac{\mu L^d}{g}} \equiv \gamma$

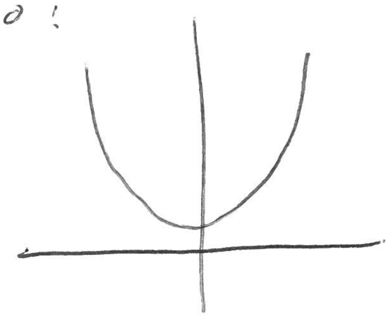
This only fixes $|\psi_0|$, there is an arbitrary <u>phase</u>!

$\psi_0 = \gamma e^{i\phi}$, $\phi \in [0, 2\pi]$ are all solutions.

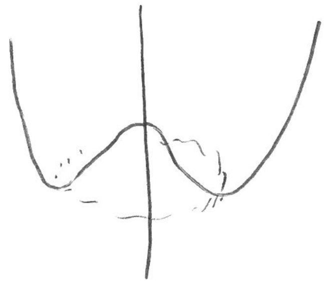
Which 'is the right one'?

We cannot look at 'quadratic' fluctuations, because changing ϕ does <u>not</u> change the action!

$\nu < 0$:



$\nu > 0$



<u>Spontaneous symmetry breaking:</u>

Assume the action ~~is~~ has a global symmetry h:

$\psi_i \to h\psi_i$, where $S[\psi] = S[h\psi]$

* Heisenberg ferromagnet: $h \in O(3)$, $\vec{S}_i \to h\vec{S}_i$

* phonons described by displacement field $\vec{u}_i$ (in ∞ system)
  $\vec{u}_i \to \vec{u}_i + \vec{a}$ where $\vec{a}$ is a lattice vector.

* Bose gas: $\psi_0 \to e^{i\phi}\psi_0$

  ↳ the state ~~ie~~ $e^{i\phi}\psi_0$ itself is not invariant under $U(1)$, it translates to another state!

  So, $S[h\psi_0] = S[\psi_0]$, but $h\psi_0 \neq \psi_0$

Ground state does <u>not</u> have same symmetry as the action!

This also happens classically:

classical ferromagnet: $Z = \mathrm{tr}\, e^{-\beta(H - \vec{h}\cdot\sum_i S_i)}$ mag. field

$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$

If we first send $|\vec{h}| \to 0$, and then $N \to \infty$, we have a rot. invariant system.

But, if we have a macroscopic system in a field, we should 1st send $N \to \infty$, followed by $h \to 0$.

In this case, the spins are alligned w/ $\vec{h}$, and so no O(3) sym. Sending $|h| \to 0$, one remains in the state w/ broken symmetry. Random fluctuation will not cause all the spins to rotate (due to entropy)

Ground state is fixed.

If a mag. material is cooled down, through the phase transition, the system picks a direction randomly (or forms domains w/ random directions).

In addition one has goldstone modes, slow fluctuation w/ arb. low energy.

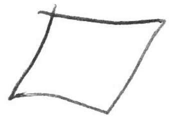
Topological phases

It is not so easy to give a good definition of a topological phase.

Normal phase: characterised by a local order parameter

Topological phase: non local properties are important.

↳ does not depend on the geometry of the system, but only on the topology

Ex: deg. of the ground state depends on genus of the 2d ~~sy~~ surface (plane  vs. torus )

1d: open system vs. periodic system.

Start w/ 1d system first: Majorana chain.

System of polarised fermions (so spinless), hopping on a 1-d lattice, with a pairing term.

[Not so easy to get polarised fermions in a real system!]

~~H =~~ Consider the open case first!

$$H = \sum_{j=1}^{N-1} -t\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right) + \left(\Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger\right)$$

$$- \sum_{j=1}^{N} \mu\left(a_j^\dagger a_j - \frac{1}{2}\right)$$

t: hopping amplitude $\qquad$ Δ: mean field S.C term

μ: chem. pot.

Why is Δ associated w/ nearest neighbours?

a_i: fermion operator! $\qquad \{a_i, a_j^\dagger\} = \delta_{ij}$

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0.$$

The number of fermion, i.e. $\sum_i a_i^\dagger a_i \equiv \cancel{\Delta} F$ is conserved modulo 2, so $\left[(-1)^{\sum_i a_i^\dagger a_i}, H\right] = 0.$

For simplicity, we set $t = \Delta = \Delta^*$, so we get

$$H = t\sum_{j=1}^{N-1} (a_j - a_j^\dagger)(a_{j+1} + a_{j+1}^\dagger) - \mu \sum_{j=1}^{N}\left(a_j^\dagger a_j - \frac{1}{2}\right)$$

The easiest way to solve the model is to introduce Majorana fermions:

$$a_j = (\gamma_{2j-1} + i\gamma_{2j})/2 \quad \text{or} \quad \gamma_{2j-1} = a_j + a_j^\dagger$$

$$a_j^\dagger = (\gamma_{2j-1} - i\gamma_{2j})/2 \qquad \gamma_{2j} = \frac{a_j - a_j^\dagger}{i}$$

So, we 'split up' the fermion in a real & im. part!

We have: $\gamma_k^\dagger = \gamma_k$: the γ's are real!

$$\gamma_i \gamma_j = -\gamma_j \gamma_i \quad i \neq j \quad \text{fermionic}$$

$$\gamma_i^2 = 1,$$

or: $\{\gamma_i, \gamma_j\} = 2\delta_{i,j}$

Using ~~any~~ pair of γ's, we can construct a fermion:

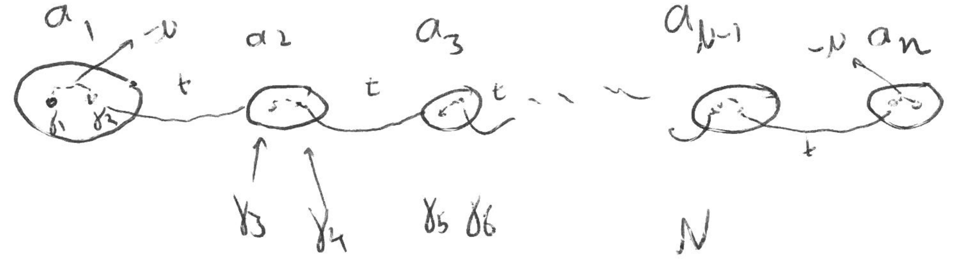
f.i. $f = (\gamma_j + i\gamma_k)/2, \quad f^\dagger = (\gamma_j - i\gamma_k)/2$

In terms of the γ's:

$$H = t\sum_{j=1}^{N-1} (i)\gamma_{2j}\gamma_{2j+1}$$

$$-\mu \sum_{j=1}^{N} (i)\gamma_{2j-1}\gamma_{2j}$$

$$a_j^\dagger a_j = \frac{1}{4}(\gamma_{2j-1} - i\gamma_{2j})(\gamma_{2j-1} + i\gamma_{2j})$$
$$= \frac{1}{4}(\gamma_{2j-1}^2 + i\gamma_{2j-1}\gamma_{2j} - i\gamma_{2j}\gamma_{2j-1} + \gamma_{2j}^2)$$
$$= \frac{1}{2} + i\gamma_{2j-1}\gamma_{2j}$$



Assume $t = 0$: then $H = -\mu \sum_{j=1}^{N} i\gamma_{2j-1}\gamma_{2j}$

$$= -\mu \sum_{j=1}^{N} \left(a_j^\dagger a_j - \frac{1}{2}\right)$$

So, GS is ~~given by~~ such that each fermionic state a_j is filled ($\mu > 0$) or empty ($\mu < 0$)

Trivial insulator

For $\nu = 0$, we instead have:

1st & last majorana are unpaired.

So we introduce fermions

$$b_1 = (\gamma_2 + i\gamma_3)/2, \; b_2 = (\gamma_4 + i\gamma_5)/2, \; \dots \; b_{N-1} = (\gamma_{2N-2} + i\gamma_{2N-1})/2$$

and we have

$$H = t\sum_{j=1}^{N-1} (i)\, \gamma_{2j}\, \gamma_{2j+1} = t\sum_{j=1}^{N-1} \left(b_j^\dagger b_j - \tfrac{1}{2}\right)$$

This hamiltonian does <u>not</u> depend on γ_1 and γ_{2N}.

So, the fermion mode $f = (\gamma_1 + i\gamma_{2N})/2$ can be empty or filled, w/o changing the energy.

So, gs is 2-fold degenerate!

The fermion mode is split into 2 γ's one on the ~~left left part of the~~ left edge, one on the right edge! They are called Majorana Bound States, and ~~(?)~~ most likely have been observed experimentally!