# Path integral approach to SC.

We start from the Euclidian action

(drop fields for the moment).

$$S[\bar{\psi},\psi] = \int_0^\beta d\tau \int d^d r\, \bar{\psi}_\sigma \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi_\sigma - g\, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow$$

(all the fields at $\psi(r,\tau)$).

How do we deal w/ quartic term?

$$e^{-a/2\, x^2} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-y^2/2a - ixy}\, dy$$

so $x^2$ in exp became linear!

Hubbard-Stratonovich:

$$1/g \int D[\bar{\Delta},\Delta]\, e^{-\int d^d r \int d\tau\, \bar{\Delta} \frac{1}{g} \Delta} = 1 \quad \text{(basic gaussian integral)}$$

So: $$e^{g\int \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int D[\bar{\Delta},\Delta]\, e^{-\int d\tau \int d^d r \left[\frac{1}{g}|\Delta|^2 - (\bar{\Delta}\psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow)\right]}$$

(complete square, and shift field $\Delta \to \Delta - g\psi_\downarrow\psi_\uparrow$)

The partition function becomes:

$$Z = \int D[\bar{\psi},\psi]\, D[\bar{\Delta},\Delta]\, e^{-S[\bar{\psi},\psi,\bar{\Delta},\Delta]}$$

where

$$S = \bar{\psi}_\sigma \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_\sigma + \frac{1}{g} |\Delta|^2 - \bar{\psi}_\downarrow \bar{\Delta} \psi_\uparrow - \bar{\psi}_\uparrow \Delta \bar{\psi}_\downarrow$$

This looks like mean-field description but

$\Delta(r, \tau)$ is a fluctuating field $\Delta \sim \psi_\downarrow \psi_\uparrow$,

with $\Delta(\beta) = \Delta(0)$, so bosonic.

We write this in terms of a Nambu spinor: $\bar{\Psi} = (\bar{\psi}_\uparrow, \psi_\downarrow)$

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}.$$

and get:

$$S = \frac{1}{g} |\Delta|^2 - \bar{\Psi} \hat{g}^{-1} \Psi, \text{ where}$$

$$\hat{g}^{-1} = \begin{bmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta \\ \bar{\Delta} & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{bmatrix} = \begin{bmatrix} [g_0^p]^{-1} & \Delta \\ \bar{\Delta} & [g_0^h]^{-1} \end{bmatrix}$$

$g_0^p$: particle greens function, $g_0^h$: hole GF.

hole: particle that moves backwards in time!

This action is quadratic in $\Psi$, so we can integrate it out:

$$Z = \int D[\bar{\Delta}, \Delta] \, e^{-\frac{1}{g} \int d\tau d^d r \, |\Delta|^2 + \operatorname{tr} \ln (\hat{g}^{-1})}$$

Here, we used $\ln\det\hat{A} = \mathrm{tr}\ln\hat{A}$;

go to diagonal basis: $\ln\det\hat{A} = \sum_a \ln \varepsilon_a = \mathrm{tr}\ln(\hat{A})$
($\varepsilon_a$ e.v. of $\hat{A}$)

What did we do? Using HS we decoupled the quartic term, and got a fluctuating field $\Delta$, integrated out the fermions. So far, we did not assume $\Delta$ is a mean ~~fi~~ field.

To continue, let's find the stationary solution, to find saddle ~~poi~~ point eqns: $\frac{\delta S}{\delta \Delta} = 0$.

First term: $\frac{\delta}{\delta\Delta}\left(-\frac{|\Delta|^2}{g}\right) = -\frac{\bar{\Delta}}{g}$

Second term: use $\partial_x \mathrm{tr} f(\hat{A}) = \mathrm{tr} f'(\hat{A})\partial_x\hat{A}$

$$= \partial_x \sum_n \frac{f^{(n)}(0)}{n!}\mathrm{tr}\hat{A}^n = \sum_n \frac{f^{(n)}(0)}{n!}\mathrm{tr}\left[(\partial_x\hat{A})\hat{A}^{n-1} + \hat{A}(\partial_x\hat{A})\hat{A}^{n-2} + \dots + \hat{A}^{n-1}(\partial_x\hat{A})\right]$$

$$= \sum_n \frac{n}{n!} f^{(n)}(0)\,\mathrm{tr}\left[\hat{A}^{(n-1)}\partial_x\hat{A}\right] = \mathrm{tr} f'(\hat{A})\partial_x\hat{A}$$

So, we find; $\frac{\delta}{\delta\Delta}\mathrm{tr}\ln \cancel{\hat{A}}\hat{\mathcal{G}}^{-1} = \mathrm{tr}\,\cancel{\hat{A}}\frac{1}{\hat{\mathcal{G}}^{-1}}\frac{\delta}{\delta\Delta}\hat{\mathcal{G}}^{-1}$

$\delta/\delta\Delta \ \hat{g}^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and we go to fourier space,

assuming $\Delta(\vec{r},\tau) = \Delta_0$ ;

$$\hat{g}^{-1} = \begin{pmatrix} i\omega_n - \frac{p^2}{2m} + \mu & \Delta_0 \\ \bar{\Delta}_0 & i\omega_n + \frac{p^2}{2m} - \mu \end{pmatrix} \qquad \xi_p = \frac{p^2}{2m} - \mu$$

So $$\hat{g} = \frac{-1}{\omega_n^2 + \xi_p^2 + |\Delta_0|^2} \begin{bmatrix} i\omega_n + \xi_p & \Delta_0 \\ -\bar{\Delta}_0 & i\omega_n - \xi_p \end{bmatrix}$$

$$\frac{\delta S}{\delta \Delta} = 0 = -\frac{1}{g}\bar{\Delta}_0 + \mathrm{tr}\left[\frac{-1}{\omega_n^2 + \xi_p^2 + |\Delta_0|^2} \begin{pmatrix} 0 & i\omega_n + \xi_p \\ 0 & -\bar{\Delta}_0 \end{pmatrix}\right]$$

which gives: $$\frac{1}{g} = \frac{T}{L_0^d} \sum_{\vec{p},n} \frac{1}{\omega_n^2 + \xi_p^2 + |\Delta_0|^2}$$

After doing the Matsubara sum and $\vec{p}$ integral: (Ex)

$$\frac{1}{g\nu} = \int_0^{\omega_D} d\xi \, \frac{\tanh\left(\lambda(\xi)/2T\right)}{\lambda(\xi)} \qquad \lambda(\xi) = \left(\xi^2 + |\Delta_0|^2\right)^{\frac{1}{2}}$$

Ginzburg Landau:

At the phase transition, $\Delta$ is small, ($\Delta \ll T$),
so we try a series expansion of the action:

Expand $\mathrm{tr}\ln \mathcal{G}^{-1}$ in powers of $\Delta$:

$$\mathcal{G}_0^{-1} = \mathcal{G}^{-1}\Big|_{\Delta=0} = \begin{pmatrix} \mathcal{G}_p^{-1} & 0 \\ 0 & \mathcal{G}_h^{-1} \end{pmatrix} \; ; \; \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}$$

So: $$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \hat{\Delta} \; , \; \mathrm{tr}\ln \mathcal{G}^{-1} = \mathrm{tr}\ln\left[\mathcal{G}_0^{-1}(1+\mathcal{G}_0\hat{\Delta})\right]$$
$$= \mathrm{tr}\ln \mathcal{G}_0^{-1} + \mathrm{tr}\ln(1+\mathcal{G}_0\hat{\Delta})$$

free energy of free el. gas
constant.

We expand the second term:

$$\mathrm{tr}\ln(1+\mathcal{G}_0\hat{\Delta}) = -\sum_{n=0}^{\infty} \frac{1}{2n}\,\mathrm{tr}(\mathcal{G}_0\hat{\Delta})^{2n}$$ (odd powers: tr is zero!)

The second order contr. of the action (see AS for details):

$$S^{(2)} = \sum_q \Gamma_q^{-1}\,|\Delta(q)|^2 \qquad \Gamma_q^{-1} = \frac{1}{g} - \frac{T}{L^d}\sum_p \mathcal{G}_{0,p}\,\mathcal{G}_{0,-p+q}$$

single particle GF.

We see, that the coefficient can ~~α~~ change sign!

Need higher order for stability!

Including gradient term at $2^{nd}$ order:

$$S_{GL} = \beta \int d^d r \left[ \frac{r(T)}{2} |\Delta|^2 + \frac{c}{2} |\partial \Delta|^2 + u|\Delta|^4 \right]$$

$r(T)$ changes sign at the phase transition, so $r \propto (T - T_c)$

One can show that higher order terms are small, and positive

$r > 0$: action minimal for $\Delta = 0$

$r < 0$ action " " $|\Delta_0| \sim \sqrt{\frac{-r}{4u}} \sim \sqrt{T_c(T_c - T)}$,

So we can write: $\Delta \sim \Delta_0 e^{2i\theta}$ $\theta$: phase of the order parameter.

$2\theta$: because $\Delta$ is ~~contains~~ fermion bi-linear

The phase is related to a Goldstone mode:

low energy excitation ~~are~~ have $\Delta_0$, but change the phase.

We are interested in response to fields:

Use minimal coupling: $p \to p - qA$, w/

$q$ charge of particle. For electrons: $p \to p + eA$,

~~where~~ where $e$ is positive

In the action, we substitute:

$$\partial_\tau \to \partial_\tau - ie\phi \;;\quad -i\vec{\nabla} \to -i\vec{\nabla} + e\vec{A}$$

scalar pot

So, we take:

$$S = \int d\tau \int d^d x \left[ \bar{\psi}_\sigma \left( \partial_\tau - ie\phi + \frac{1}{2m}(-i\vec{\nabla} + e\vec{A})^2 - \nu \right) \psi_\sigma - g\, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right].$$

This is invariant under:

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi} \quad \left( \text{so } \Delta \to e^{2i\alpha}\Delta \right.$$
$$\left. \bar{\Delta} \to e^{-2i\alpha}\bar{\Delta} \right)$$

$$e\phi \to e\phi - \partial_\tau \alpha$$

$$e\vec{A} \to e\vec{A} - \vec{\nabla}\alpha$$

So, the GL action should also be invariant!

This can be achieved by covariant derivatives:

$$-i\vec{D}\Delta = (-i\vec{\nabla} + 2e\vec{A})\Delta$$

$$D_\tau \Delta = (\partial_\tau - 2ie\phi)\Delta$$

Assuming that the variation in $\Delta_0$ is small (neglect $\vec{\nabla}\Delta_0$ & $\partial_\tau \Delta_0$):

$$(-i\vec{\nabla} + 2e\vec{A})\Delta_0 e^{2i\theta} = 2\left[ (\vec{\nabla}\theta) + 2e\vec{A} \right] \Delta_0 e^{2i\theta}$$

$$(\partial_\tau - 2ie\phi)\Delta_0 e^{2i\theta} = 2i\left[ (\partial_\tau \theta) - 2e\phi \right] \Delta_0 e^{2i\theta}$$