Fermi liquid theory: very brief overview

Explains why the 'free electron model' works so well in describing metals, even though Coulomb interaction is typically not negligible.

Free el. gas at $T=0$: fill all single particle states up to ε_F.

$\rightarrow$ momentum distribution function $n_0(p) = \theta(p - p_f)$,

or at finite T: $n_\beta(p) = \dfrac{1}{e^{\beta(\varepsilon(p)-\varepsilon_F)}+1}$

Landau's idea: lets turn on interactions adiabatically,

* one-to-one correspondence between excitations of free & interacting system.

So, in the latter case, there is also a Fermi-surface!

* excitation can be labeled by same quantum numbers: charge, spin, momentum.

Excitations: quasi-particles are comb. of many electrons, or 'the electrons are dressed', weakly interacting

quasiparticles can decay, but their lifetime becomes very long close to Fermi surface (small phase space).

In addition, the effective mass,

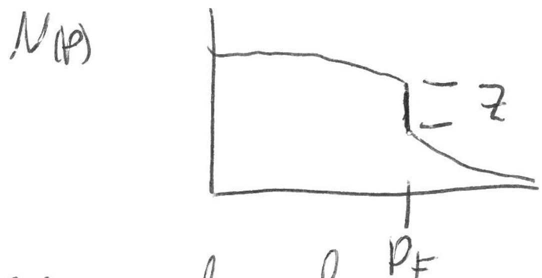
$V_F = \frac{P_F}{m^*} \neq \frac{P_f}{m_e}$ is different from the free case,

This can be observed in $c_V = \frac{m^* p_F^2}{3} k_B T$,

and for 'heavy fermion' can be large $100 \sim 1000$

(UPt_3, $CeCu_6$)

The momentum distribution in GS: states at $p > p_F$ can be occupied, even at ~~T~~ $T=0$: there is a 'step' at p_F: fermi surface.



So, there are low lying excitation (quasi-particle, quasi-hole)

Fermi liquid is stable, as long as interactions are repulsive.

However, attractive interactions are possible, leading to ~~an~~ a super-conducting state!

Origin of attractive interaction:
electron-phonon coupling.

Time ~~t~~ scale on which an e^- perturbs an ~~ion~~ ion
$\sim E_F^{-1}$; time scale for an ion to relax: $\omega_D^{-1} \gg E_F^{-1}$
(ω_D = Debye frequency)
So, a second e^- sees a slightly different pot. for a long time $\rightarrow$ attractive interaction, but only for energies within ω_D of fermi energy.
The attraction leads to the formation of 'Cooper pairs', ~~2~~ e^- with opp. momentum, in a singlet.

Some general properties:

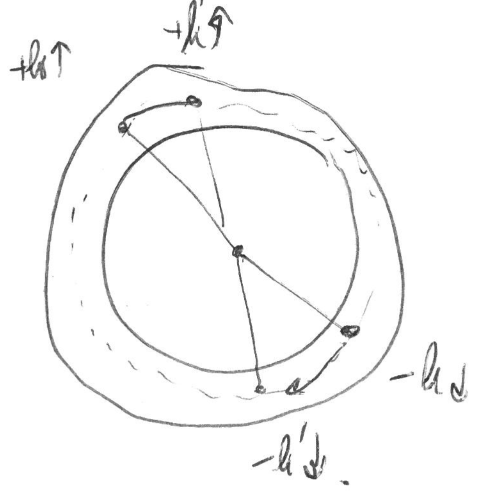
* Persistent currents, w/o. voltage drop (as long as $I < I_c$)
* Meissner effect: external magnetic fields are expelled, as long as $H < H_c$
* Energy gap Δ to excitations above GS (in BCS case).
* In some SC's (type II): vortices can be present if $H_{c_1} < H < H_{c_2}$

Flux is $\Phi_{SC} = \frac{\Phi_0}{2} = \frac{h}{2e} \rightarrow$

T_c: conv. SC: $\sim 30\,K$; ~~A~~ High T_c materials: $40 - 130\,K$.

Formation of 'Cooper pairs':

Can occur in Window ω_D/v_F around fermi energy:



~~Due to~~ There is a large phase space for scattering process from $k \to k+p = k'$ close to fermi momentum.

So the effect of even a weak attr. interaction can be appreciable.

We will describe this in terms of the BCS hamiltonian:

$$H_{BCS} = \sum_{k\sigma} \underbrace{(\epsilon_k - \mu)}_{\xi_k} c^{\dagger}_{k\sigma} c_{k\sigma} - \frac{g}{L^d} \sum_{k,k',q} c^{\dagger}_{k+q\uparrow} c^{\dagger}_{-k\downarrow} c_{-k'+q\downarrow} c_{k'\uparrow}$$

$g > 0$

The interaction makes analysis difficult, so we will use <u>mean field theory</u> first, then use path integral formalism to study SC's in presence ~~So~~ of fields.

Goal of mean-field theory: gap equation.

Assumption: $|\Omega_s\rangle$ (GS) contains Cooper pairs:

$$\Delta = \frac{g}{L^d} \sum_k \langle \Omega_s | c_{-k\downarrow} c_{k\uparrow} | \Omega_s \rangle \neq 0$$

$$\rightsquigarrow \bar{\Delta} = \frac{g}{L^d} \sum_k \langle \Omega_s | c^+_{k\uparrow} c^+_{-k\downarrow} | \Omega_s \rangle \neq 0.$$

So, $|\Omega_s\rangle$ can <u>not</u> be a state w/ definite number of particles.

Δ takes role of 'order parameter': $\Delta = 0$ in normal state
$\Delta \neq 0$ in SC.

In a mean field treatment (assuming on-shell scattering, $q=0$)

$$c^+_{k\uparrow} c^+_{-k\downarrow} = \langle c^+_{k\uparrow} c^+_{-k\downarrow} \rangle + \underbrace{(c^+_{k\uparrow} c^+_{-k\downarrow} - \langle\ \rangle)}_{\delta^+_k}$$

we write: $c^+_{k\uparrow} c^+_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \simeq$

$$c^+_{k\uparrow} c^+_{-k\downarrow} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle + \langle c^+_{k\uparrow} c^+_{-k\downarrow} \rangle c_{-k'\downarrow} c_{k'\uparrow}$$

$$- \langle c^+_{k\uparrow} c^+_{-k\downarrow} \rangle \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$$

(up to order δ, drop δ^2)

This gives

$$H^{MF}_{BCS} = \sum_k \xi_k c^+_{k\sigma} c_{k\sigma} - \left(\bar{\Delta} c_{-k\downarrow} c_{k\uparrow} + \Delta c^+_{k\uparrow} c^+_{-k\downarrow} \right) + \frac{L^d}{g} |\Delta|^2$$

We see that H does change # of electrons!

We can use a Bogoliubov transform to diagonalize.

So: $\psi_k^\dagger = (c_{k\uparrow}^\dagger, c_{-k\downarrow})$ $\quad \psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$

If we assume Δ is real (otherwise, we can do a 'basis transform' to make it real), we get

$$H = \sum_k \psi_k^\dagger \begin{bmatrix} \xi_k & -\Delta \\ -\Delta & -\xi_k \end{bmatrix} \psi_k + \sum_k \xi_k + \frac{L^d |\Delta|^2}{g}$$

To get rid of Δ terms:

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$$

Note $\alpha_{k\uparrow}^\dagger = \cos\theta_k \, c_{k\uparrow}^\dagger + \sin\theta_k \, c_{-k\downarrow}$, comb. of a particle and hole, w/ def. spin and momentum!

We need: $\tan 2\theta_k = -\Delta / \xi_k$, or

$\sin 2\theta_k = -\Delta/\lambda_k$, $\cos 2\theta_k = \xi_k/\lambda_k$, with $\lambda_k = (\Delta^2 + \xi_k^2)^{\frac{1}{2}}$,

and we obtain

$$H = \sum_{k\sigma} \lambda_k \alpha_{k\sigma}^\dagger \alpha_{k\sigma} + \sum_k (\xi_k - \lambda_k) + \frac{L^d}{g}\Delta^2$$

The elementary excitations ~~are cont~~ are created of $\alpha_k^\dagger$ and have a minimal energy Δ, the gap

At low T, it is hard to create excitation,

To find the GS $|\Omega_s\rangle$, we need a state that is anihilated by all $\alpha_{k\sigma}$:

$$|\Omega_s\rangle = \prod_k (\alpha_{k\uparrow}\alpha_{k\downarrow})\underbrace{|\Omega\rangle}_{\text{vacuum of } c_{k\uparrow}, c_{k\downarrow}} = \prod_k (\cos\theta_k - \sin\theta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|\Omega\rangle$$

We can now solve for the gap selfconsistently:

$$\Delta = \frac{g}{L^d}\sum_k \langle\Omega_s| c_{-k\downarrow} c_{k\uparrow}|\Omega_s\rangle$$

$$= -\frac{g}{L^d}\sum_k \sin\theta_k \cos\theta_k = \frac{g}{2L^d}\sum_k \frac{\Delta}{(\Delta^2+\xi_k^2)^{\frac{1}{2}}}$$

Transform the sum to an integral (~~are~~ from $-\omega_D$ to ω_D):

$$\Delta \approx \frac{g\Delta}{2}\int_{-\omega_D}^{\omega_D} \frac{\overbrace{\nu(\xi)}^{\text{density of states} \approx \nu \text{ constant.}}}{(\Delta^2+\xi^2)^{\frac{1}{2}}} d\xi$$

(where g is non-zero!)

$$= g\Delta\nu \operatorname{arcsinh}(\omega_D/\Delta) \quad \text{so:}$$

$$\Delta \simeq \frac{\omega_D}{\sinh(1/g\nu)} \underset{g\nu \ll 1}{\sim} 2\omega_D e^{-1/g\nu}$$

Even for small g, the value for Δ $\cancel{g_0}$ can be big if ν, the density of states is high.

Also, the form of the gap is non-perturbative in g!.

Note: g.S. energy w/ $g > 0$, is <u>lower</u> than the g.S energy of system w/ $g = 0$. (free fermions).