Free energy of the electron gas: perturbation theory:

~~1st order~~ $F = -T \ln Z$,

$$Z = \int D(\bar{\psi},\psi)\, e^{-S[\bar{\psi},\bar{\psi}]}$$

, where

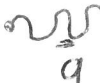
$$S[\bar{\psi},\psi] = \sum_p \bar{\psi}_p \left(-i\omega_n + \frac{|\vec{p}|^2}{2m} - \mu\right)\psi_p$$

$$- \frac{1}{2L^3} \sum_{p,p',q} \bar{\psi}_{p+q}\, \bar{\psi}_{p'-q}\, V(q)\, \psi_{p'}\, \psi_p$$

1st order contribution:

$$\Gamma^{(1)} = \frac{1^2}{2L^3} \sum_{p,p',q} \langle \bar{\psi}_{p+q}\, \bar{\psi}_{p'-q}\, V(q)\, \psi_{p'}\, \psi_p \rangle_0$$

Feynman rules:

* Interaction: 

* Contraction: $\langle \psi_p \bar{\psi}_p \rangle_0 = \mathcal{G}_{0,p} = \frac{1}{-i\omega_n + \frac{|\vec{p}|^2}{2m} - \mu}$

* Sum over all 4-momenta at vertex is zero

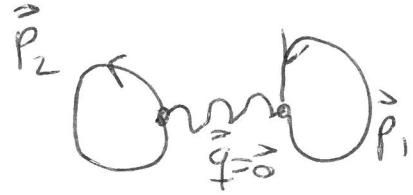
* Grassman var. anti commute: sign for each independent loop.

* Consider only <u>connected</u> diagrams (we're calculating <u>$\ln Z$</u>).

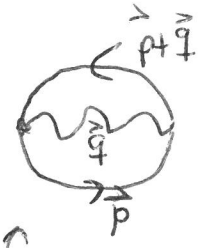
$$V(\vec{q}) = \frac{e^2}{q^2}, \quad V(z) = \frac{e}{z}, \text{ or:}$$

$$V(z,\lambda) = \frac{e}{z} e^{-\lambda z} \rightsquigarrow V(q) = \frac{e^2}{q^2+\lambda^2}$$

There are two contributions at 1st order,



Hartree



Fock

Hartree term: the Green's functions measure the electron density:

$$\rightsquigarrow \propto \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{i\omega_n - \frac{|\vec{p}|^2}{2m} + \mu}$$

$$= \beta \int \frac{d^3p}{(2\pi)^3} n_F(\vec{p}) = \beta N,$$

↑ Matsubara sum

which is the total charge. # The Hartree contribution cancels against a homogeneous background charge (not always the case!)

Hartree term: 'classical' density-density interaction.

[drop diagrams w/ $\vec{q}=\vec{0}$ interaction line]

The Fock contribution: (note: 1 fermion loop!)
(spin: factor of 2)

$$-\frac{T^2}{L^3}\sum_{p,q} G_{\omega,p}\, G_{\omega,p+q}\, V(q)$$

$$= \frac{-T^2}{L^3}\sum_{p,p'} G_{\omega,p}\, G_{\omega,p'}\, V(p'-p)$$

$(q = p'-p)$

$$= \frac{1}{L^3}\sum_{\vec{p},\vec{p}'} n_F(\varepsilon_{\vec{p}})\, n_F(\varepsilon_{\vec{p}'})\, \frac{e^2}{|\vec{p}-\vec{p}'|^2}$$

$$T\sum_{i\omega_n} G_{\omega p} \equiv n_F(\varepsilon_p)$$

Evaluating this at zero temperature ($n_F \to$ step function)

$$F^{(1)}(T=0) = -\frac{1}{L^3}\sum_{\varepsilon_p,\varepsilon_{p'}<\mu} \frac{e^2}{|\vec{p}-\vec{p}'|^2}$$

$$= \frac{-e^2 L^3}{(2\pi)^4} p_f^4 \;;\quad \text{Power of } p_f\text{: } p_F^{3+3-2}$$

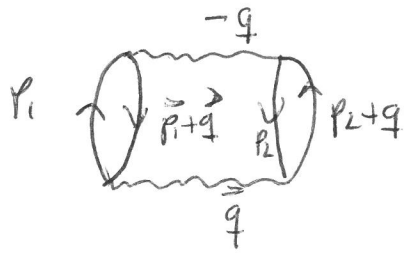
Constant & evaluate integrals (use Legendre polynomials).

The contribution is negative, because the electrons avoid one another!

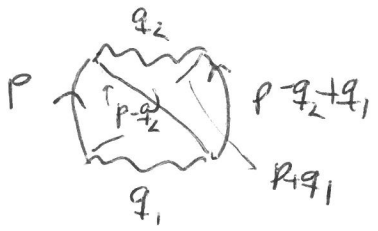
There is a problem, however! $F^{(0)} + F^{(1)}$ give rise to a, divergences at ε_F, and an ρ that is zero. Reason: long range Coulomb interaction.
Sol'n: screening or higher order PT.

2nd order terms (drop the ones w/ $\vec{q}=0$ lines)

(a)



(b)



$$F^{2,a} = -\frac{T^3}{L^6}\sum_{p_1,p_2,q} G_{\sigma,p_1} G_{\sigma,p_1+q} G_{\sigma,p_2} G_{\sigma,p_2+q} V(q)^2$$

$$F^{2,b} = +\frac{T^3}{2L^6}\sum_{p,q_1,q_2} G_{\sigma,p} G_{\sigma,p+q_1} G_{\sigma,p+q_1-q_2} G_{\sigma,p-q_2} V(q_1) V(q_2)$$

b cont.: one particle hole pair.
a cont. two ind. particle hole pairs

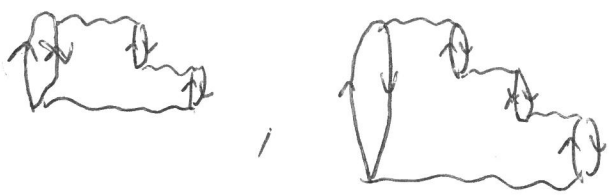
How do these contribute: $\frac{1}{-i\omega_n + \xi(p) - \mu}$: suppressed away from $|\vec{p}| = p_F$

In (a) this means that $|\vec{q}|$ is small

(b) $|\vec{q}_1|, |\vec{q}_2|$ small.

In a, there are two independent momentum integals, in b only one. So, there is a much larger phasespace for (a), which will dominate.

At higher order, we find that 'ring' diagrams dominate:



, etc.

These diagrams correspond to:

$$F_{RPA}^{(n)} = \frac{-T}{2n}(-1)^n \sum_q \left(\left(-\frac{2T}{aL^3}\right) V(q) \sum_p G_{\sigma p} G_{\sigma, p+q} \right)^n$$

$2n$: Combinatorics!

$(-1)^n$: fermion loop

↳ $\frac{(n-1)!}{n!}$ → # ways the int. can be arranged in a ring.

↳ exponent

Important class of diagrams, which have large phase space!

We can actually sum over all of them:

$$F_{RPA} = \sum_{n=1}^{\infty} F_{RPA}^{(n)} = \frac{T}{2} \sum_q \ln\left(1 - V(q)\,\Pi_q\right),$$

where $\Pi_q = \frac{2T}{L^3} \sum_p G_{\sigma p} G_{\sigma, p+q}$ is the <u>polarization operator</u>, related to screening:

$$V_{eff} = \frac{1}{V(q) - \Pi_q} = \frac{V(q)}{\varepsilon(q)}, \quad \text{where } \varepsilon(q) = 1 - V(q)\Pi_q$$

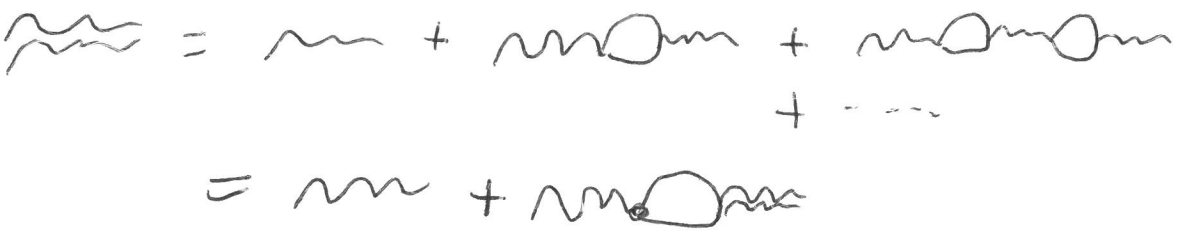
is a gen. dielectric function.

We see this from comparing $N^{(1)} = -\partial_\mu F^{(1)}$ and $N_{RPA} = -\partial_\mu F_{RPA}$:

$$N^{(1)} = -\frac{2T^3}{L^3} \sum_{p,q} \left(G_{\sigma p}\right)^2 G_{p+q}\, V(q)$$

$$N_{RPA} = \frac{T}{2} \sum_q \frac{V(q)\,\partial_\mu \Pi_q}{1 - V(q)\Pi_q} = \ldots = -\frac{2T^2}{L^3} \sum_q V_{eff}(q) \sum_p G_{p+q} \left(G_p\right)^2$$

We can represent V_{eff} as  :



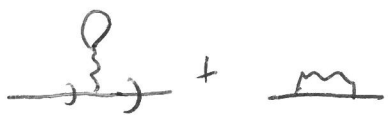
So, symbolically

$$\rightsquigarrow \quad \text{(wavy)} = \frac{\text{(wavy)}}{1 - V(q)\Pi_q} = \frac{V(q)}{1 - V(q)\Pi_q}$$

We can do something similar for the self-energy (ϕ^4 theory):

Write the greens function as:

where is 'one particle irreducible':



$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \cdots$$

$$= G_0 + G_0 \Sigma G \qquad \text{Dyson eq-n.}$$

$$\left[\text{(double arrow)} = \text{(single arrow)} + \text{(single arrow with blob)} \right]$$

In momentum space: $G_p = G_{0,p} + G_{0,p} \Sigma_p G_p$, or

$$G_p = \left[1 - G_{0,p}\Sigma_p\right]^{-1} G_{0,p} = \frac{1}{G_{0,p}^{-1} - \Sigma_p} = \frac{1}{p^2 + m^2 - \Sigma_p}$$

So, to find greens function, we need to compute the self energy!