ϕ^4 theory:

$$Z = \int D\varphi \, e^{-S[\varphi]} \qquad S[\varphi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m\varphi^2}{2} + g\phi^4\right]$$
$$= S_0[\varphi] + S_{in}[\varphi]$$

We are interested in expectation values:

$$C_n = \langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \frac{\int D\varphi \, e^{-S[\varphi]} \varphi(x_1) \cdots \varphi(x_n)}{\int D\varphi \, e^{-S[\varphi]}}$$

$\langle \quad \rangle_0$: calculated w.r.t. $S_0[\varphi]$ instead.

$C_2(x_1, x_2) = G(x_1 - x_2)$ is the Greens function, or propagator.

$\langle \varphi(x_1)\varphi(x_2)\rangle_0 = G_0(x_1 - x_2)$: free propagator. Ex: calculate it here!

P.T.: write $e^{-S[\varphi]} = e^{-S_0[\varphi]} e^{-S_{int}[\varphi]}$, and expand the latter.

$$\langle X[\varphi] \rangle \sim \frac{\sum_n \frac{(-g)^n}{n!} \langle X[\varphi] \left(\int d^d y \, \varphi^4(y)\right)^n \rangle_0}{\sum_n \frac{(-g)^2}{n!} \langle \left(\int d^d y \, \varphi^4(y)\right)^n \rangle_0}$$

collection of fields!

Apply this to greens function and look at 1st order.

$$\mathcal{G}^{(1)}(x,x') = -g\Big[\langle \varphi(x) \Big(\int d^d y\, \varphi^4(y)\Big)\, \varphi(x')\rangle_0 - \langle \varphi(x)\varphi(x')\rangle_0 \langle \Big(\int d^d y\, \varphi^4(y)\Big)\rangle_0 \Big]$$

With Wick's theorem, we obtain 15 terms for 1st contribution: (drop integrals, etc)

$$\langle \varphi(x)\, \varphi^4(y)\, \varphi(x')\rangle_0 = 3 \langle \varphi(x)\varphi(x')\rangle_0 \langle \varphi(y)\varphi(y)\rangle_0^2 + 12 \langle \varphi(x)\varphi(y)\rangle_0 \langle \varphi(y)\varphi(y)\rangle_0 \langle \varphi(y)\varphi(x')\rangle_0$$

$$\rightsquigarrow (-g)\int \Big[3\, \mathcal{G}^0(x-x')\, \mathcal{G}^0(0)^2 + 12\, \mathcal{G}_0(x-y)\, \mathcal{G}_0(0)\, \mathcal{G}^0(y-x')\Big]$$

subtraction: $-3\, \mathcal{G}^0(x-x')\big(\mathcal{G}^0(0)\big)^2$.

The number of terms grows quickly, 2nd order cont. from ~~numerator~~ numerator 945 terms !

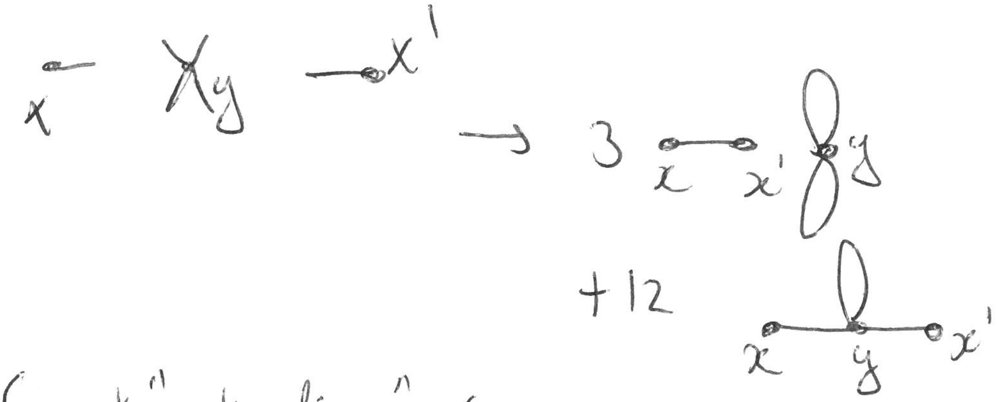
Need good bookkeeping device!

For each variable, write a 

, and each vertex: 

, and combine in all possible ways.

Lines become free green functions:



Connect "outer lines" (—∘$_x$) first.

Diagrams of 'one piece': connected diagrams

Vaccum diagrams: have an interaction vertex that does not connect to any external line.

Contributions w/ vaccum diagram ~~These~~ cancell due to expansion of denominator!

\# of contractions of 2n objects:

arbitrary order: $(2n)!$;
ordering in a pair irrelevant: $\frac{1}{2^n}$
ordering of pairs irrelevant: $\frac{1}{n!}$

$$\frac{(2n)!}{2^n n!} = (2n-1)(2n-3)\cdots 1 \equiv (2n-1)!!$$

Greens function etc. only depends on $(x-y)$, so we can/should go to momentum rep:

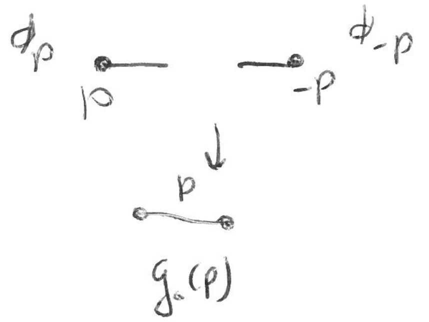
$S_0[\phi] = \frac{1}{2} \sum_p \phi_p (p^2+m^2) \phi_{-p}$, which gives

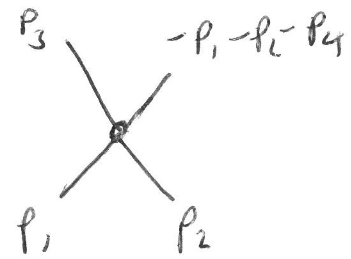
$$g_{0,p} = \langle \phi_p \phi_{-p} \rangle_0 = \frac{1}{p^2+m^2}$$

So, we can only contract vertices w/ opp. momentum! (Momentum conservation!)

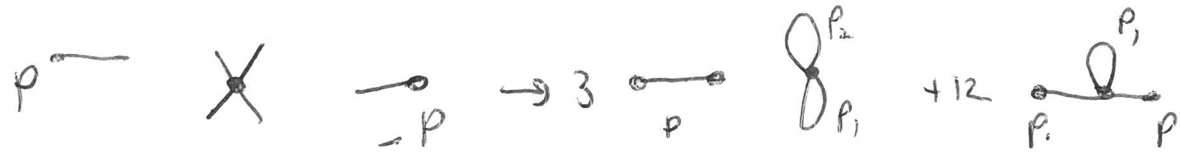
Interaction piece:

$$\int d^d y\, \phi(y)^4 \rightsquigarrow \frac{1}{L^d} \sum_{p_1 .. p_4} \phi_{p_1} \phi_{p_2} \phi_{p_3} \phi_{p_4}\, \delta_{p_1+p_2+p_3+p_4,0}$$





Sum of 'momenta in' is zero!



Integrals over internal momenta!

Application to ground state energy of the electron gas.

Strength of the interaction: 'space' occ. by e^- : r_0^3

Uncertainty rel: $\rightsquigarrow$ $T_{min} \sim \hbar^2/m r_0^2$

Coulomb : e^2/r_0

$$\frac{e^2}{r_0}\frac{m r_0^2}{\hbar^2} = \frac{r_0}{a_0} = r_s$$

$\hookrightarrow$ Bohr rad. $= \hbar^2/e^2 m$

High density r_s small: weak interactions

Typically: $r_s \sim 3 - 5$ intermediate strength.

Nevertheless, PT works very well!

Free energy: $F = -T \ln Z$, where $Z = \int D(\psi,\bar\psi) e^{-S[\psi,\bar\psi]}$

$\uparrow$ Grassmann!

$$S[\psi,\bar\psi] = \sum_p \bar\psi_p\left(-i\omega_n + \frac{|\vec p|^2}{2m} - \mu\right)\psi_p - \frac{T}{2L^3}\sum_{p,p',q} \bar\psi_{p+q}\bar\psi_{p'-q} V(\vec q)\psi_{p'}\psi_p$$

here, p is 4 momentum $p = (\omega_n, \vec p)$, we dropped spin!

Look at non-interacting gas first (can cal. integral over $\mathcal{D}(\bar\psi_p,\psi_p)$):

$$F^{(0)} = -T \ln\left[\prod_p \beta\left(-i\omega_n + \frac{|\vec p|^2}{2m} - \mu\right)\right]$$

Doing the matsubara sum first gives

$$F^{(0)} = -T \sum_{\vec{p}} \log\left[1 + e^{-\beta\left(\frac{\vec{p}^2}{2m} - \mu\right)}\right]$$

$$= \sum_{\vec{p}} \ln\left[1 + e^{-\beta\left(\frac{p^2}{2m} - \mu\right)}\right]^{-T}$$

$$\underset{T\to 0}{=} \sum_{\frac{|\vec{p}|^2}{2m} - \mu < 0} \left(\frac{|\vec{p}|^2}{2m} - \mu\right)$$

One does the sum by transforming it to an integral:

$$F^0 \to \overset{\text{spin}}{2} L^3 \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{\vec{p}^2}{2m} - \mu\right)$$

$$= 2L^3 \, 4\pi \int_0^{p_F} \frac{dp}{(2\pi)^3} p^2 \left(\frac{p^2}{2m} - \mu\right) = \frac{L^3}{\pi^2}\left(\frac{1}{5} p_F^3 \left(\frac{p_F^2}{2m}\right) - \frac{1}{3} p_F^3 \mu\right)$$

$$p_F : \frac{p_F^2}{2m} = \mu \quad p_F = \sqrt{2m\mu} \qquad = \frac{L^3}{\pi^2} \mu \left(\frac{-2}{15}\right) p_F^3$$

$$N = 2L^3 \int_{|\vec{p}| \le p_F} \frac{d^3p}{(2\pi)^3} 1 = \frac{L^3}{\pi^2} \frac{1}{3} p_F^3 \qquad \left(n = \frac{N}{V} = \frac{1}{3\pi^2} p_F^3\right)$$

So: $F^{(0)} = -\frac{2}{5} \mu N$