In CM, we are interested in correlation functions at finite T and ρ. These can be calculated from the (grand-canonical) Partition function:
(or the log of...)

$$Z = \sum_n \langle n | e^{-\beta(\hat{H} - \mu \hat{N})} | n \rangle \qquad \beta = \frac{1}{k_B T}$$

This looks very similar to

$\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle$, namely, we write

$$e^{-\beta(\hat{H}-\mu\hat{N})} = e^{-i/\hbar\,(\hat{H}-\mu\hat{N})(-i\underset{\beta\hbar}{\underset{\shortparallel}{\tau}})}$$

So, if we set $t = -i\tau = -i\beta\hbar$, the statistical operator is like an 'imaginary time' evolution, and we can apply the 'coherent-state path integral':

We use $d(\bar{\psi},\psi) = \begin{cases} \prod_i d\bar{\psi}_i\, d\psi_i/\pi & \text{bosons} \\ \prod_i d\bar{\psi}_i\, d\psi_i & \text{fermions} \end{cases}$
$\hookrightarrow$ Grassmann

$$\mathbb{1} = \int d(\bar{\psi},\psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} |\psi\rangle\langle\psi|$$

Use this, to get rid of summation over Fock space:

$$Z = \int d(\bar{\psi}, \psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \sum_n \langle n | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle$$

$\hat{H}$ and $\hat{N}$: contain an even number of $a^\dagger, a$'s.

For fermions, we pick up a sign if we commute the two scalar products:

$$Z = \int d(\bar{\psi}, \psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \sum_n \langle \overset{\overset{-1 \text{ fer}}{1 \text{ bos}}}{\downarrow}\!\!\zeta \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \langle n | \psi \rangle$$

$$= \int d(\bar{\psi}, \psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \langle \zeta \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi \rangle .$$

Recipe: * divide β in N steps $\beta/N = \Delta\beta$;
* expand to lin. order in $\Delta\beta$
* insert set of coherent states at each step

$$Z = \int d(\bar{\psi}, \psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \langle \zeta \psi | \left[e^{-\Delta\beta \hat{H}} e^{\Delta\beta \mu \hat{N}} \right]^N | \psi \rangle + O(\Delta\beta^2)$$

$$\simeq \int d(\bar{\psi}, \psi)\, e^{-\sum_i \bar{\psi}_i \psi_i} \langle \zeta \psi | \left[\int d(\bar{\psi}', \psi')\, e^{-\sum_i \bar{\psi}'_i \psi'_i} | \psi' \rangle \langle \psi' | e^{-\Delta\beta \hat{H}} e^{\Delta\beta \mu \hat{N}} \right]^N | \psi \rangle$$

If $\hat{N} = \sum_i a_i^\dagger a_i$ and $\hat{H}$ commute, this is exact $\left[\text{f.i. } \hat{H} = \sum_{ij} a_i^\dagger h_{ij} a_j + \sum_{ijkl} V_{ijkl}\, a_i^\dagger a_j^\dagger a_k a_l \right]$

Note: in H, a^+ are on the left, a's on the right ~ normal ordering, so we can evaluate the expectation values!

ψ_i^n ← time slice, ← site index ; $\psi^n = \{\psi_i^n\}$

We obtain:

$$Z = \int_{\substack{\bar\psi^0 = \zeta\bar\psi^N \\ \psi^0 = \zeta\psi^N}} \prod_{n=0}^{N-1} d(\bar\psi^n, \psi^n)\, e^{-\Delta\beta \sum_{n=0}^{N-1} \left[\frac{\bar\psi^n - \bar\psi^{n+1}}{\Delta\beta} \cdot \psi^n + H(\bar\psi^{n+1}, \psi^n) - \mu N(\bar\psi^{n+1}, \psi^n) \right]}$$

where $H(\bar\psi, \psi') = \sum_{i,j} \bar\psi_i h_{ij} \psi_j + \sum V_{ijkl} \bar\psi_i \bar\psi_j \psi_k \psi_l$

$$= \frac{\langle \psi | \hat{H}(a^+, a) | \psi' \rangle}{\langle \psi | \psi' \rangle}$$

and similar for $N(\bar\psi, \psi')$.

We take cont. limit:

$$Z = \int D(\bar\psi, \psi)\, e^{-S[\bar\psi, \psi]}$$ and same Boundary cond!

$$D(\bar\psi,\psi) = \lim_{N\to\infty} \prod_{n=1}^{N} d(\bar\psi^n, \psi^n)$$

$$S[\bar\psi, \psi] = \int_0^\beta d\tau \left[\bar\psi \partial_\tau \psi + H(\bar\psi, \psi) - \mu N(\bar\psi, \psi) \right].$$

Z is written in the time representation.

It's usefull to go to frequency space!

$$\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \psi_n \, e^{-i\omega_n \tau}$$

$$\psi_n = \frac{1}{\sqrt{\beta}} \cancel{\Sigma} \int_0^\beta d\tau \, \psi(\tau) \, e^{i\omega_n \tau}$$

where: $\omega_n = \begin{cases} 2n\pi T & \text{bosons} \\ (2n+1)\pi T & \text{fermions} \end{cases}$ $\quad n \in \mathbb{Z}.$

Matsubara frequencies

$$Z = \int \underbrace{D(\bar\psi, \psi)}_{= \prod_n d(\bar\psi_n, \psi_n)} \, e^{-\beta S[\bar\psi, \psi]}$$

$$S = \sum_{ij,n} \bar\psi_{in} \left[(-i\omega_n - \mu)\delta_{ij} + h_{ij} \right] \psi_{jn}$$
$$+ \frac{1}{\beta} \sum_{ijkl, n_i} V_{ijkl} \, \bar\psi_{in_1} \bar\psi_{jn_2} \psi_{kn_3} \psi_{ln_4} \, \delta_{n_1+n_2, n_3+n_4}$$

where we used $\int_0^\beta dt \, e^{-i\omega_n \tau} = \beta \, \delta_{\omega_n, 0}$

Example: let's calculate Z (free energy) for a free gas.

$$H = H_0 = \sum_{ij} \bar\psi_i h_{ij} \psi_j = \sum_a \bar\psi_a \varepsilon_a \psi_a$$

Action: $S = \sum_{a,n} \bar\psi_{an} (-i\omega_n + \varepsilon_a - \mu) \psi_{an}$

$\varepsilon_a - \mu = \xi_a$

So, we have $Z = \prod_a Z_a$,

Where $Z_a = \int D(\bar{\psi}_a, \psi_a)\, e^{-\beta \sum_{n,a} \bar{\psi}_{an}(-i\omega_a + \xi_a)\psi_{an}}$

$$= \prod_n \left[\beta(-i\omega_n + \xi_a)\right]^{-\zeta} \quad \leftarrow \pm \text{ bos, ferm.}$$

(Gaussian integration!)

The free energy!

$$F \equiv -T \ln Z = T\zeta \sum_{a,n} \ln\left[\beta(-i\omega_n + \xi_a)\right]$$

So, we have the ferm. $\text{Sum} = \sum_n h(\omega_n)$

We calculate this by using a function $g(z)$, w/ poles at $z = i\omega_n$, and do $\oint g(z) h(-iz)$.

$$g(z) = \begin{cases} \dfrac{\beta}{e^{\beta z} - 1} & \text{bos} \quad (+1) \\ \dfrac{\beta}{e^{\beta z} + 1} & \text{ferm} \quad (-1) \end{cases}$$

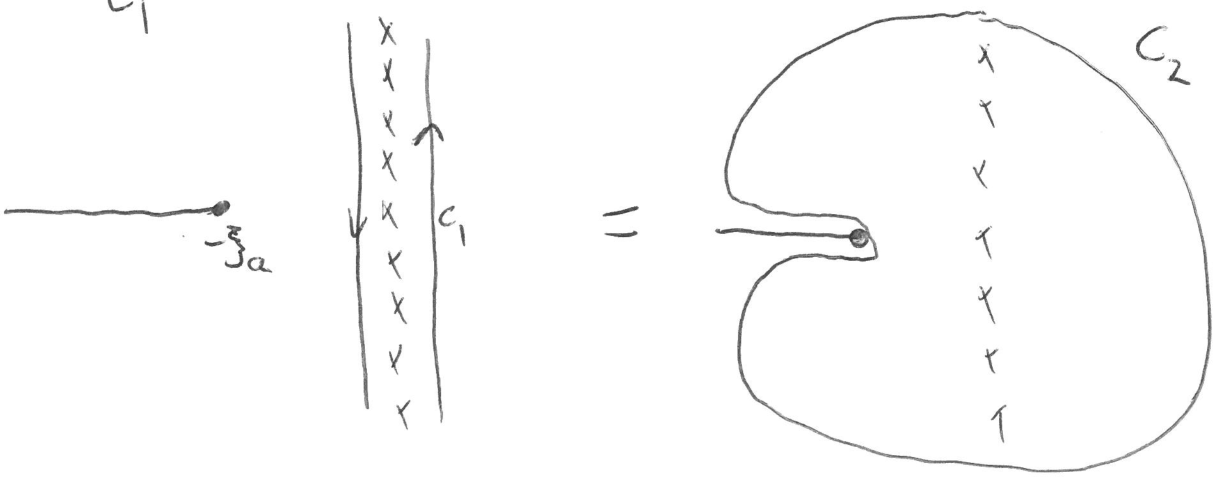
↑ res. at $i\omega_n$.

Note: distribution function for Bosons & fermions!

$$\frac{\zeta}{2\pi i} \oint_{C_1} dz\, g(z) h(-iz) = \zeta \sum_n \operatorname{Res}\left(g(z) h(-iz)\right)\Big|_{z = i\omega_n}$$

$$= \sum_n h(\omega_n) = \text{Sum}$$

C_1 in our case:



In the case at hand:

$$h(\omega_n) = \zeta T \ln[\beta(-i\omega_n + \xi_a)]$$

$$= \zeta T \ln(\beta(i\omega_n - \xi_a)) + \text{const.}$$

Sum becomes: (great circle does not contribute)

$$\text{Sum} = \frac{T}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)\left(\ln[\varepsilon + i\delta - \xi_a] - \ln[\varepsilon - i\delta - \xi_a]\right)$$

Using $g(\varepsilon) = \zeta \frac{d}{d\varepsilon} \ln(1 - \zeta e^{-\beta\varepsilon})$ and

$$\lim_{\delta \to 0^+} \int_{-\infty}^{\infty} \frac{f(x)}{x \pm i\delta}\, dx = \mp i\pi f(0) + P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx$$

We get: $F = \zeta T \sum_a \ln\left[1 - \zeta e^{-\beta(\varepsilon_a - \mu)}\right]$,

the expected result!

In perturbation theory, we want to calculate effect of weak interactions.

Ex $$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 - g x^4}, \text{ assume } g \ll 1,$$

So we write: $I(g) \sim \sum_n g^n I_n$,

$$g^n I_n = \frac{(-g)^n}{n!} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} x^{4n} \underset{n \gg 1}{\sim} \left(-\frac{16 g n}{e}\right)^n$$

?

Stirling $n! \sim n^n e^{-n}$

Even for small g, there is an n so that this becomes large!

For small g, we can however app. this ~~by~~ well by taking finite number of terms!

~~Pertur~~ Even then, the expressions we get can contain all kinds of UV or IR divergencies, which one has to deal with!

ϕ^4 theory:

$$Z = \int \mathcal{D}\varphi \, e^{-S[\varphi]} \qquad S[\varphi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m\varphi^2}{2} + g\,\phi^4 \right]$$

We are interested in expectation values:

$$C_n = \langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \frac{\int \mathcal{D}\varphi \, e^{-S[\varphi]} \varphi(x_1) \cdots \varphi(x_n)}{\int \mathcal{D}\varphi \, e^{-S[\phi]}}$$

$\langle \quad \rangle_0$: calculated w.r.t. $S_0[\varphi]$ instead.

$C_2(x_1, x_2) = g(x_1 - x_2)$ is the Greens function, or propagator.

$\langle \varphi(x_1)\varphi(x_2)\rangle_0 = g_0(x_1 - x_2)$: free propagator. Ex: calculate it here!

P.T.: write $e^{-S[\phi]} = e^{-S_0[\varphi]} e^{-S_{int}[\varphi]}$, and expand the latter.

$$\langle X[\varphi] \rangle \sim \frac{\sum_n \frac{(-g)^n}{n!} \langle X[\varphi] \left(\int d^d y \, \phi^4(y) \right)^n \rangle_0}{\sum_n \frac{(-g)^2}{n!} \langle \left(\int d^d y \, \varphi^4(y) \right)^n \rangle_0}$$

collection of fields!

Apply this to greens function and look at 1st order.

$$\mathcal{G}^{(1)}(x,x') = -g\Big[\langle \varphi(x)\Big(\int d^dy\, \varphi^4(y)\Big)\varphi(x')\rangle_0$$
$$- \langle \varphi(x)\varphi(x')\rangle_0 \langle \Big(\int d^dy\,\varphi^4(y)\Big)\rangle_0\Big]$$

With Wick's theorem, we obtain 15 terms for 1st contribution:

$$\langle \varphi(x)\,\varphi^4(y)\,\varphi(x')\rangle_0 = 3\,\langle \varphi(x)\varphi(x')\rangle_0 \langle \varphi(y)\varphi(y)\rangle_0^2$$
$$+ 12 \langle \varphi(x)\varphi(y)\rangle_0 \langle \varphi(y)\varphi(y)\rangle_0 \langle \varphi(y)\varphi(x')\rangle_0$$

$$\rightsquigarrow -g\int\Big[3\,\mathcal{G}^0(x-x')\,\mathcal{G}^0(0)^2 + 12\,\mathcal{G}_0(x-y)\,\mathcal{G}_0(0)\,\mathcal{G}(y-x')$$

subtraction: $-3\,\mathcal{G}^0(x-x')\big(\mathcal{G}^0(0)\big)^2$.

The number of terms grows quickly, 2nd order

Cont. from ~~numerator~~ numerator 945 terms !

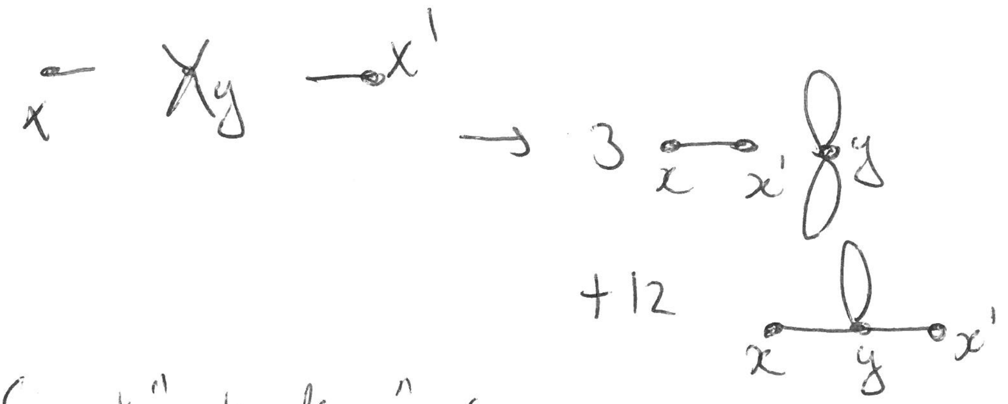
Need good bookkeeping device!

For each variable, write a , and each

vertex: , and combine in

all possible ways.

Lines become free green functions:



Connect "outer lines" (x) first.

Diagrams of 'one piece': connected diagrams

Vacuum diagrams: have an interaction vertex that does not connect to any external line.

These cancell ~~m~~ due to expansion of denominator!