Path integrals in quantum mechanics, one body system.

Classical physics: path of a particle is obtained by minimising the action (energy)

Quantum: all path contribute to amplitudes

[E not strictly conserved]

One particle: $\hat{H}(\hat{p},\hat{q}) = \hat{T}(\hat{p}) + \hat{V}(\hat{q})$, with

often: $\hat{T} = \frac{\hat{p}^2}{2m}$

Amplitude ~~for a~~ to start in $|q_i\rangle$ $(t=0)$ and end up in $|q_f\rangle$ (t):

$A = \langle q_f| e^{-i\hat{H}t/\hbar} |q_i\rangle$, which we can split up in N parts as $A = \langle q_f| \left[e^{-i\hat{H}\Delta t/\hbar}\right]^N |q_i\rangle$, where $\Delta t = t/N$

We split exponential, to first order in Δt:

$$e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} + O(\Delta t)^2,$$ so we write:

$$\langle q_f| \mathbb{1}_N e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} \mathbb{1}_{N-1} \cdots \mathbb{1}_1 e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} |q_i\rangle,$$

where $\mathbb{1}_n = \int dq_n\, dp_n\, |q_n\rangle\langle q_n|p_n\rangle\langle p_n|$, and

$$\langle q|p\rangle = \langle p|q\rangle^* = e^{ipq/\hbar}/\sqrt{2\pi\hbar}$$

We act w/ the operators on their eigenstates:

$$|q_1\rangle\langle q_1|p_1\rangle\langle p_1| e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} |q_0\rangle \qquad (q_0 = q_i)$$

$$= e^{-iT(p_1)\Delta t/\hbar} e^{-iV(q_0)\Delta t/\hbar} \langle q_1|p_1\rangle\langle p_1|q_0\rangle\, |q_1\rangle$$

$$= \frac{1}{(2\pi\hbar)} e^{-i\Delta t/\hbar\left[T(p_1)+V(q_0) - \frac{p_1}{\Delta t}(q_1-q_0)\right]} |q_1\rangle$$

In total:

$$A = \int_{\substack{q_0=q_i \\ q_N=q_f}} \frac{dp_N}{2\pi\hbar} \prod_{n=1}^{N-1} \left(\frac{dp_n}{2\pi\hbar}\right) dq_n\, e^{-\frac{i\Delta t}{\hbar}\sum_{n=0}^{N-1}\left[V(q_n) + T(p_{n+1}) - \frac{p_{n+1}}{\Delta t}(q_{n+1}-q_n)\right]}$$

This is a $2N-1$ dim. integral over classical phase space!

$(x_n = (q_n, p_n)$ parametrise the paths

Only those path for ~~of~~ which the exp. varies smoothly contribute, otherwise phases cancel.

We can take cont. limit: $N\to\infty$, with $t = \Delta t N$ fixed

$$\Delta t \sum_{n=0}^{N-1} \to \int_0^t dt' \; ; \quad \frac{(q_{n+1} - q_n)}{\Delta t} \to \partial_{t'} q\big|_{t'=t_n} \equiv \dot{q}\big|_{t'=t_n}$$

Also: $V(q_n) + T(p_{n+1}) \to T(p|_{t'=t_n}) + V(q|_{t'=t_n}) = H(x|_{t'=t_n})$,

i.e. the classical Hamiltonian!

The measure is written as

$$\lim_{N\to\infty} \frac{1}{(2\pi\hbar)^N} \int_{\substack{q_N = q_f \\ q_0 = q_i}} dp_N \prod_{n=1}^{N-1} dq_n dp_n = \int_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}x$$

So: $$A = \langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int_{\substack{q(t)=q_f \\ q(0)=q_i}} \mathcal{D}x \; e^{i/\hbar \int_0^t dt' [p\dot{q} - H(p,q)]}$$

This is the Hamilton formulation of the P.I.

What is the Lagrangian formulation?

We can do the integral over p, because it's quadratic.

Re discretize, and use Gaussian integration:

$$\int d\vec{v}\, e^{-\frac{1}{2}\vec{v}^T \cdot A \cdot \vec{v} + \vec{J}\cdot\vec{v}} = \sqrt{\frac{(2\pi)^N}{\det A}}\, e^{\frac{1}{2}\vec{J}^T \cdot A^{-1} \cdot \vec{J}},$$

with $A = 1/m\, \mathbb{1}$, $\vec{J} = \dot{\vec{q}}$.

Result:

$$A = \int \mathcal{D}q\, e^{i/\hbar \int_0^t dt' \left[\frac{m\dot{q}^2}{2} - V(q)\right]}$$

$\hookrightarrow$ classical Lagrangian!

and $$\mathcal{D}q = \lim_{N\to\infty} \left(\frac{-iNm}{2\pi\hbar t}\right)^{N/2} \prod_{n=1}^{N-1} dq_n$$

P.I. recipe:

* discretise time in N steps.
* split the exponential, to 1st order in Δt
* insert 'correct' resolution of $\mathbb{1}$
* evaluate $\langle | | \rangle$'s
* take cont. limit.

We can do the same for partition functions in quantum stat. sys:

$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$, where Tr is a sum over all states

$= \int dq \, \langle q | e^{-\beta \hat{H}} | q \rangle$

This resembles $\langle q_f | e^{-i \hat{H} t/\hbar} | q_i \rangle$, if one sets $t = -i\beta\hbar$, so we Wick rotate to imaginary time: $t \to -i\tau$, and evolve in im. time: $e^{-it\hat{H}/\hbar} \to e^{-\tau \hat{H}/\hbar}$.

Path integrals in field theory are very much the same, but we do not integrate over $(0+1)$ dims, but a $(d+1)$ dim 'surfaces', i.e. over $\phi(x)$ at each timestep.

What is the 'correct' resolution of $\mathbb{1}$?

We need the eigenstates of a_i, $a_i^\dagger$, in terms of which $\hat{H}$ is written.

Coherent states (bosons).

Generic state in Fock space: $|\phi\rangle = \sum_{\{n_i\}} c_{n_1,n_2,\ldots} |n_1, n_2, \ldots\rangle$
has contributions with varying particle number.

One can not write an eigenstate of a^+!
[if $|\phi\rangle$ has at least n_0 particles, $a^+|\phi\rangle$ has at least n_0+1],

so we look for a $|\phi\rangle$, s.t. $\underline{a_i |\phi\rangle = \phi_i |\phi\rangle}$

$|\phi\rangle = e^{\sum_i \phi_i a_i^+} |0\rangle$ does the job!
$\{\phi_i\}$ is a collection of complex numbers.

To show that $a_i|\phi\rangle = \phi_i|\phi\rangle$, one Taylor expands, and uses $[a_i, (a_j^+)^n] = n (a_i^+)^{n-1} \delta_{ij}$

Taking herm. conj. gives:

$\underline{\langle\phi| a_i^+ = \langle\phi| \bar{\phi}_i}$, and $\langle\phi| = \langle 0| e^{\sum_i \bar{\phi}_i a_i}$

where $\bar{\phi}_i$ is c.c. of ϕ_i

Also, $a_i^+ |\phi\rangle = \partial_{\phi_i} |\phi\rangle$

The overlap of coherent states:

$$\langle\theta|\phi\rangle = \langle 0| e^{\sum_i \bar{\theta}_i a_i} |\phi\rangle = e^{\sum_i \bar{\theta}_i \phi_i} \langle 0|\phi\rangle$$
$$= e^{\sum_i \bar{\theta}_i \phi_i}$$

The coherent states form an overcomplete set of states:

$$\mathbb{1}_{\mathcal{F}} = \int \prod_i \frac{d\bar{\phi}_i \, d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| ,$$

where $d\bar{\phi}_i \, d\phi_i = d\,\mathrm{Re}\,\phi_i \, d\,\mathrm{Im}\,\phi_i$ are <u>independent</u>

For fermions, we need to take into account that the $\{a_i\}$ <u>anti</u> commute, so we need Grassmann variables: $\eta_i \eta_j = -\eta_j \eta_i$;

So: $\eta^2 = 0$.

Functions of η: $f(\eta) = f(0) + f'(0)\eta$, no higher powers!

Differentiation: $\partial_{\eta_i} \eta_j = \delta_{ij}$, and ∂_{η_i} <u>anti</u> commutes!

Integration: $\int d\eta \, f(\eta) = \int d\eta \, (f(0) + f'(0)\, \eta) = f'(0) = \partial_\eta f(\eta)$

$[\int d\eta_i = 0 ; \int d\eta_i \, \eta_i = 1]$

We also require: $\{ \eta_i , a_j \} = 0$

We then have: $|\eta\rangle \equiv e^{-\sum_i \eta_i a_i^{\dagger}} |0\rangle$, ~~so~~ with

$a_i |\eta\rangle = \eta_i |\eta\rangle$

Properties are similar to bosonic case:

$$\langle \eta | = \langle 0 | e^{-\sum_i a_i \bar{\eta}_i} = \langle 0 | e^{\sum_i \bar{\eta}_i a_i},$$

but $\bar{\eta}_i$ <u>is not</u> some comp. conj. of η_i, they are completely independent.

Also, $\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} = 1$, so there are <u>no</u> factors of π in resolution of $\mathbb{1}$:

$$\int \prod_i d\bar{\eta}_i\, d\eta_i\, e^{-\sum_i \bar{\eta}_i \eta_i} |\eta\rangle\langle\eta| = \mathbb{1}_F$$