Tutorial on ODEs

September 6, 2019

1 Motion of a particle in a constant electromagnetic field

Consider a particle of mass m, charge q, initial position $\boldsymbol{r}_0 = \boldsymbol{0}$ and initial velocity $\boldsymbol{v}_0$. Gravity will be neglected in this problem.

a) Determine the trajectory of the particle in a constant electric field $\boldsymbol{E} = E\,\boldsymbol{e}_x$ assuming no friction. If possible, write down the equation of this trajectory.

b) Add a friction force $\boldsymbol{f} = -\frac{m}{\tau}\,\boldsymbol{v}$ and solve the differential equation for $\boldsymbol{v}(t)$ with the initial condition $\boldsymbol{v}(t) = \boldsymbol{v}_0$. What is the speed limit ? Give a physical interpretation of the time τ.

c) Determine the trajectory of the particle in a constant magnetic field $\boldsymbol{B} = B\,\boldsymbol{e}_z$ assuming no friction.
Method 1 : Take $\boldsymbol{v}_0 = v_0\cos(\theta)\,\boldsymbol{e}_z + v_0\sin(\theta)\,\boldsymbol{e}_y$ and solve the system of differential equations for $x(t), y(t), z(t)$.
Method 2 (optional) : Show that $||\boldsymbol{v}(t)||$ and $\widehat{(\boldsymbol{v},\boldsymbol{B})}$ are invariant of motion. Then decompose the velocity as $\boldsymbol{v} = \boldsymbol{v}^{\perp} + \boldsymbol{v}^{\parallel}$ with $\boldsymbol{v}^{\perp}$ the component orthogonal to $\boldsymbol{B}$ and $\boldsymbol{v}^{\parallel}$ the component parallel to $\boldsymbol{B}$, and show that the orthogonal motion is necessarily circular.

2 Linear ODEs with constant coefficients

Find the general solution of the following ODEs :

a)

$$y''' - 2y'' + y' - 2y = 0 \tag{1}$$

b)

$$\forall n \in \mathbb{N}^{\star}, \quad \frac{d^n y}{dx^n} - y(x) = 0 \tag{2}$$

c)

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{3}$$

3 High speed motion of a sphere in a gas

At small velocities, the friction force exerted on a sphere in a gas is given by the Stokes' law which depends on viscosity and is linear in speed. However at high velocities, the resistance to the motion is mainly due to the inertia of the gas as it is pushed apart by the sphere and viscous effects are negligible. Then, one can show by dimensional analysis that the drag should be proportional the squared speed.
Write down the equation of motion of a sphere at high speed in a gas (1D, neglecting gravity) and solve it for the initial condition $v(t = 0) = v_0$.

4 Thickness distribution of sea ice

The steady distribution of sea ice thickness satisfies the following 2nd order linear ODE (cf. Assignment 1) :

$$g''(h) + \left(\frac{1}{H} - \frac{q}{h}\right)g'(h) + \frac{q}{h^2}\, g(h) = 0 \quad \text{with} \quad (q, H) \in \mathbb{R}^2 \tag{4}$$

Integrate it twice and impose the normalization condition $\int_0^{+\infty} g(h)dh = 1$.

5 An isobaric equation

$$\frac{dy}{dx} = \frac{y-x}{y+x} \tag{5}$$

a) Show that this first order ODE is isobaric, then solve it thanks to a transformation that makes it separable.

b) The solution actually corresponds to an ubiquitous curve in nature. To identify it, write (5) in polar coordinates and solve it.

6 Exact but non-separable 1st order ODEs

Check that the following differential equations are exact and find the general solution $\varphi(x, y) = cst$.

a)

$$\left[6y(x^2 + y^2 - 1)^2 + 54yx^2\right]\frac{dy}{dx} + 6x(x^2 + y^2 - 1)^2 + 54xy^2 = 0 \tag{6}$$

If $cst = 0$, the solution is an astroid : the curve one gets by rolling a circle of radius $\frac{1}{4}$ inside a circle of radius 1. How many cusps has this curve ? It is very difficult to answer this elementary question unless one is aware of the following parametrization :

$$\begin{cases} x(t) = \cos^3(t) \\ y(t) = \sin^3(t) \end{cases} \tag{7}$$

b)

$$\left[4y^3 + 4y(x^2 + a^2)\right]\frac{dy}{dx} + 4x(x^2 - a^2) + 4xy^2 = 0, \quad a \in \mathbb{R} \tag{8}$$

Show that the solution can be put on the form

$$\left[(x-a)^2 + y^2\right]\left[(x+a)^2 + y^2\right] = b^4 \tag{9}$$

which is the Cartesian equation of a Cassini oval when $b \in \mathbb{R}$. How to caracterize the set of points $P = (x, y)$ with a geometric property? It is helpful to consider the fixed points $P_1 = (a, 0)$ and $P_2 = (-a, 0)$.

7 Linear equidimensional equations

An equation is said to be equidimensional in x if the scale $x \to ax$ leaves it unchanged. One can transform it into an autonomous equation thanks to the change of variable $x = e^t$. Linear equidimensional equations are of the form :

$$\sum_{i=0}^{n} a_i x^i \frac{d^i y}{dx^i} = 0 \quad \text{with} \quad a_0, a_1, \ldots, a_n \in \mathbb{R} \tag{10}$$

a) Show that such an equation is transformed to a linear ODE with constant coefficients and that, as a counterpart, one can directly look for solutions of the form x^r.

b) Find the general solution of the following ODEs :

i)

$$x^2y''(x) - 4xy'(x) + 6y(x) = 0 \tag{11}$$

ii)

$$x^3y'''(x) + xy'(x) - y(x) = 0 \tag{12}$$

8 Three methods for a simple looking equation

Consider the following apparently simple ODE :

$$(x - y)\frac{dy}{dx} = 1 \tag{13}$$

Three methods are suggested to solve it, classified by order of subtlety.

a) Is it isobaric ? Is it exact ? If not, look for an integrating factor. *Method advocated in the course book.*

b) Note that (13) is of the form $\frac{dy}{dx} = F(v)$ with v a linear combination of x and y. *A special case to be recognized.*

c) Look at x as the dependent variable ; use a geometric property to plot the solution $y(x)$. *A very specific trick.*

9 Laudau amplitude equation

The transition to turbulence is a famous problem in the theory of hydrodynamic instability. A flow is said to be linearly stable when a small perturbation vanishes exponentially in time and it is linearly unstable when it grows exponentially in time. However, non-linear effects can have a huge influence on the stability of a flow. Let's consider a flow that becomes linearly unstable when the control parameter R reaches a critical value R_c. Laudau proposed to describe the evolution of the amplitude A of a perturbation by :

$$\frac{dA}{dt} = \sigma A - \frac{l}{2}A^3 \tag{14}$$

where $\sigma \propto R - R_c$ is the growth (or decay) rate of the perturbation and l the Landau coefficient. The non-linear term is stabilizing when $l > 0$ but destabilizing when $l < 0$.

a) What kind of non-linear ODE is it? Make a change variable $v(A)$ to get a linear inhomogenous ODE. Solve the latter for the initial condition $A(0) = A_0$ and infer the time evolution of A^2.

b) Study the limit of $A^2(t)$ when t goes to infinity depending on the signs of σ and l. Try to identify a supercritical and a subcritical bifurcation with the corresponding diagrams A_∞ versus R.

10 A Clairaut equation

Let's denote $p = \frac{dy}{dx}$. Find the general solution of the following so-called Clairaut equation :

$$y = px + \ln(p) \tag{15}$$

11 Generation of waves by the wind

For centuries, scientists and sailors have been vexed by how the wind and the sea conspire to create the waves upon the oceans. In 1957, Phillips and Miles independently proposed mechanisms for wind waves generation. Complementary rather than competitive, their theories can be combined in a single formalism from which emerges a forced harmonic oscillator with a negative damping :

$$\frac{d^2\eta}{dt^2} - 2\gamma\frac{d\eta}{dt} + \omega^2\eta(t) = p_a(t) \tag{16}$$

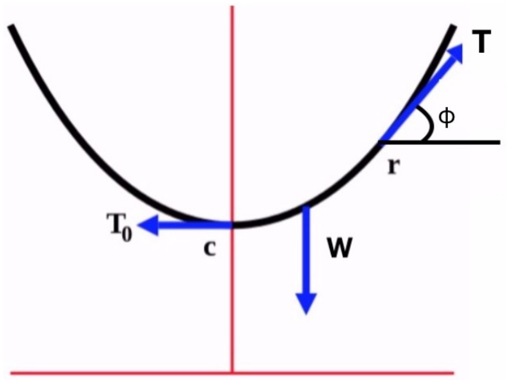

Figure 1: Force balance on a portion of a catenary

The variable η corresponds to one mode of the water surface oscillating with the frequency ω in the absence of wind. The function $p_a(t)$ models the random pressure fluctuations of the turbulent wind in the Phillips mechanism. $\gamma > 0$ is the growth rate of the waves due to the components of air pressure in phase quadrature with η in the Miles mechanism.

a) Assuming $\gamma \ll \omega$, find for an arbitrary function p_a the solution of (16) for the initial conditions $\eta(0) = 0$ and $\frac{d\eta}{dt}\big|_{t=0} = 0$.

b) Set $\gamma = 0$ and $p_a(t) = \cos(\omega t)$. What happens to the solution ? What general phenomenon is the Phillips mechanism based on ?

c) How do the waves grow when $\gamma \neq 0$? Can the Miles mechanism occur if $p_a(t) = 0$?

12 The catenary

A catenary is the curve that an idealized hanging chain assumes under its own weight when supported only at its ends. The chain is modeled by a 1D curve $y(x)$ with a mass per unit lenght λ. We assume it is inextensible and so flexible that any force of tension is tangent to it. Consider a portion of the chain of length s between its minimum point C (where the slope $\frac{dy}{dx}$ vanishes) and a point $\boldsymbol{r}$ to its right ; s is the curvilinear abscissa. At static equilibrium, the weight and the tensions at each extremity balances ; see figure 1.

a) Thanks to the force balance, show that $\frac{dy}{dx} = \frac{s}{a}$ with $a \triangleq \frac{T_0}{\lambda g}$, g being the gravitational acceleration.

b) Set $\frac{dy}{dx} \equiv p$ and derive the following first order non-linear ODE :

$$\frac{dp}{dx} = \frac{\sqrt{1+p^2}}{a} \tag{17}$$

c) Solve (17) and infer the equation of the catenary $y = y(x)$.

13 The brachistochrone curve

The word “brachistochrone” comes from the Greek terms *brakhistos*, “the shortest”, and *chronos*, “time”, so the brachistochrone curve is the curve of fastest descent. A bead slides without friction from a point A down to a point B under the action of gravity $\boldsymbol{g}$. Which path $y(x)$ should it follow to arrive in the shortest time ?

a) Let s be the curvilinear abscissa ; see figure 2. Justify that the velocity is $\frac{ds}{dt} = \sqrt{2gy}$, then show that one has to minimize the functional

$$F[y] = \int_{x_a}^{x_b} \mathcal{L}(y, y')\, dx \text{ where prime stands for derivation } \textit{w.r.t. } x \text{ and } \mathcal{L}(y, y') = \sqrt{\frac{1 + y'(x)^2}{2gy(x)}} \tag{18}$$

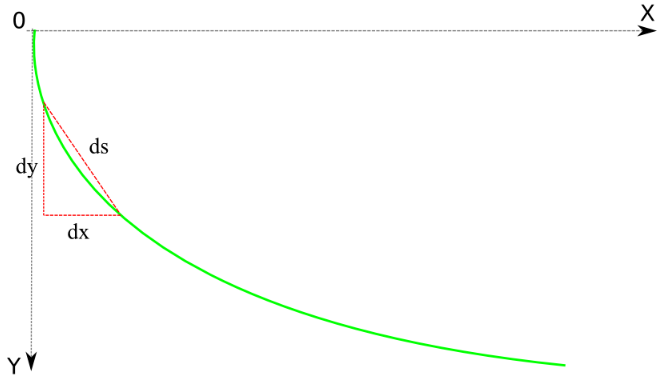

Figure 2: Brachistochrone curve

b) As the Lagrangian $\mathcal{L}$ does not depend explicitly on x, derive the Beltrami identity from the Euler-Lagrange equations of variational calculus :

$$\mathcal{L} - y' \frac{\partial \mathcal{L}}{\partial y'} = cst \tag{19}$$

c) Infer a first order differential equation of degree 2 setting $cst = D$; what is the physical interpretation of D? One can get a nice parametric solution $\left(x(\theta), y(\theta)\right)$ thanks to the change of variable $y' = \cot\left(\frac{\theta}{2}\right)$. Finding $y(\theta)$ is straightforward from the ODE then integrate the chain rule $\frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta}$ to obtain $x(\theta)$. After some trigonometric manipulations, one should recognize the parametrization of a cycloid.

14 Singularities of the hypergeometric equation

Numerous special functions can be expressed in terms of solutions of the hypergeometric equation. Study the nature of all its singularities :

$$x(1-x)\, y''(x) + [c - (a+b+1)x]\, y'(x) - ab\, y(x) = 0, \quad (a,b,c) \in \mathbb{R}^3 \tag{20}$$

15 Frobenius method 1

P3, exam 2013-11-09. Consider the following differential equation:

$$y''(x) - 2x\, y'(x) + (E-1)\, y(x) = 0, \quad E \in \mathbb{R} \tag{21}$$

derived from the quantum mechanical harmonic oscillator.

a) Use Frobenius method to find the odd and even solutions for this equation.

b) Determine the values of the energy E for which the series terminate, resulting in polynomials of finite order.

c) Write down explicitly the polynomials corresponding to the three lowest energies as obtained from your expansion and give their energies. (The units are arbitrary here.)

16 Frobenius method 2

Inspired by **P3, exam 2011-01-05.** Consider the following differential equation:

$$x(x+1)\, y''(x) + (1-x)\, y'(x) + y(x) = 0 \tag{22}$$

a) Recall Fuchs theorem and find one solution in a simple closed form via the Frobenius method.

b) Use the solution found in the previous question to infer a second solution in a series form.