FOURIER SERIES

Periodic phenomena involving waves, rotating machines (harmonic motion), or other repetitive driving forces are described by periodic functions. Fourier series are a basic tool for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic boundary conditions. Fourier integrals for nonperiodic phenomena are developed in Chapter 20. The common name for the field is **Fourier analysis**.

19.1 GENERAL PROPERTIES

A Fourier series is defined as an expansion of a function or representation of a function in a series of sines and cosines, such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{19.1}$$

The coefficients a_0, a_n, and b_n are related to $f(x)$ by definite integrals:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos ns \, ds, \quad n = 0, 1, 2, \ldots, \tag{19.2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin ns \, ds, \quad n = 1, 2, \ldots, \tag{19.3}$$

which are subject to the requirement that the integrals exist. Note that a_0 is singled out for special treatment by the inclusion of the factor $\frac{1}{2}$. This is done so that Eq. (19.2) will apply to all a_n, $n = 0$ as well as $n > 0$.

The conditions imposed on $f(x)$ to make Eq. (19.1) valid are that $f(x)$ have only a finite number of finite discontinuities and only a finite number of extreme values (maxima and minima) in the interval $[0, 2\pi]$.[1] Functions satisfying these conditions may be

[1] These conditions are **sufficient** but not **necessary**.

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called **piecewise regular**. The conditions themselves are known as the **Dirichlet conditions**. Although there are some functions that do not obey these conditions, they can be considered pathological for purposes of Fourier expansions. In the vast majority of physical problems involving a Fourier series, the Dirichlet conditions will be satisfied.

Expressing $\cos nx$ and $\sin nx$ in exponential form, we may rewrite Eq. (19.1) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{19.4}$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \quad n > 0, \tag{19.5}$$

and

$$c_0 = \frac{1}{2} a_0. \tag{19.6}$$

Sturm-Liouville Theory

The ODE

$$-y''(x) = \lambda y(x)$$

on the interval $[0, 2\pi]$ with boundary conditions $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$ is a Sturm-Liouville problem, and these boundary conditions make it Hermitian. Therefore its eigenfunctions, either $\cos nx$ $(n = 0, 1, \dots)$ and $\sin nx$ $(n = 1, 2, \dots)$, or $\exp(inx)$ $(n = \dots, -1, 0, 1, \dots)$, form a complete set, with eigenfunctions of different eigenvalues orthogonal. Since the eigenfunctions have respective values n^2, those of different $|n|$ will automatically be orthogonal, while those of the same $|n|$ can be orthogonalized if necessary. Defining the scalar product for this problem as

$$\langle f|g\rangle = \int_0^{2\pi} f^*(x) g(x)\, dx,$$

it is easy to check that $\langle e^{inx}|e^{-inx}\rangle = 0$ for $n \neq 0$, and if we write $\cos nx$ and $\sin nx$ as complex exponentials, it is also easy to see that $\langle \sin nx|\cos nx\rangle = 0$. To make the eigenfunctions normalized, a simple approach is to note that the average value of $\sin^2 nx$ or $\cos^2 nx$ over an integer number of oscillations is 1/2 (again for $n \neq 0$), so

$$\int_0^{2\pi} \sin^2 nx\, dx = \int_0^{2\pi} \cos^2 nx\, dx = \pi \quad (n \neq 0),$$

and $\langle e^{inx}|e^{inx}\rangle = 2\pi$.

The relationships identified above indicate that the eigenfunctions $\varphi_n = e^{inx}/\sqrt{2\pi}$, $(n = \dots, -1, 0, 1, \dots)$ form an orthonormal set, as do

$$\varphi_0 = \frac{1}{\sqrt{2\pi}}, \quad \varphi_n = \frac{\cos nx}{\sqrt{\pi}}, \quad \varphi_{-n} = \frac{\sin nx}{\sqrt{\pi}}, \quad (n = 1, 2, \dots),$$

so expansions in these functions have the forms given in Eqs. (19.1) to (19.3) or Eqs. (19.4) to (19.6). Since we know that the eigenfunctions of a Sturm-Liouville operator form a complete set, we know that our Fourier-series expansions of L^2 functions will at least converge in the mean.

Discontinuous Functions

There are significant differences between the behavior of Fourier- and power-series expansions. A power series is essentially an expansion about a point, using only information from that point about the function to be expanded (including, of course, the values of its derivatives). We already know that such expansions only converge within a radius of convergence defined by the position of the nearest singularity. However, a Fourier series (or any expansion in orthogonal functions) uses information from the entire expansion interval, and therefore can describe functions that have "nonpathological" singularities within that interval. However, we also know that the representation of a function by an orthogonal expansion is only guaranteed to converge **in the mean**. This feature comes into play for the expansion of functions with discontinuities, where there is no unique value to which the expansion must converge. However, for Fourier series, it can be shown that if a function $f(x)$ satisfying the Dirichlet conditions is discontinuous at a point x_0, its Fourier series evaluated at that point will be the arithmetic average of the limits of the left and right approaches:

$$f_{\text{Fourier series}}(x_0) = \lim_{\varepsilon \to 0} \left[\frac{f(x_0+\varepsilon) + f(x_0-\varepsilon)}{2} \right]. \tag{19.7}$$

For proof of Eq. (19.7), see Jeffreys and Jeffreys or Carslaw (Additional Readings). It can also be shown that if the function to be expanded is continuous but has a finite discontinuity in its first derivative, its Fourier series will then exhibit uniform convergence (see Churchill, Additional Readings). These features make Fourier expansions useful for functions with a variety of types of discontinuities.

Example 19.1.1 SAWTOOTH WAVE

An idea of the convergence of a Fourier series and the error in using only a finite number of terms in the series may be obtained by considering the expansion of

$$f(x) = \begin{cases} x, & 0 \le x < \pi, \\ x - 2\pi, & \pi < x \le 2\pi. \end{cases} \tag{19.8}$$

This is a sawtooth wave form, as shown in Fig. 19.1. Using Eqs. (19.2) and (19.3), we find the expansion to be

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots \right]. \tag{19.9}$$

Figure 19.2 shows $f(x)$ for $0 \le x < 2\pi$ for the sum of 4, 6, and 10 terms of the series. Three features deserve comment.

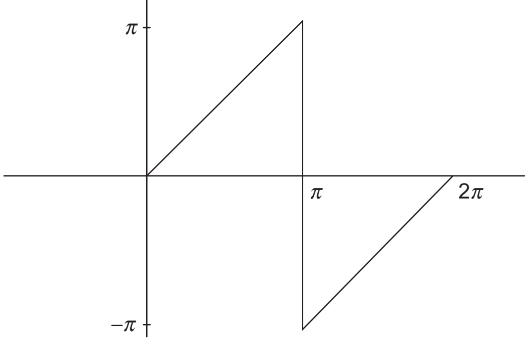

FIGURE 19.1 Sawtooth wave form.

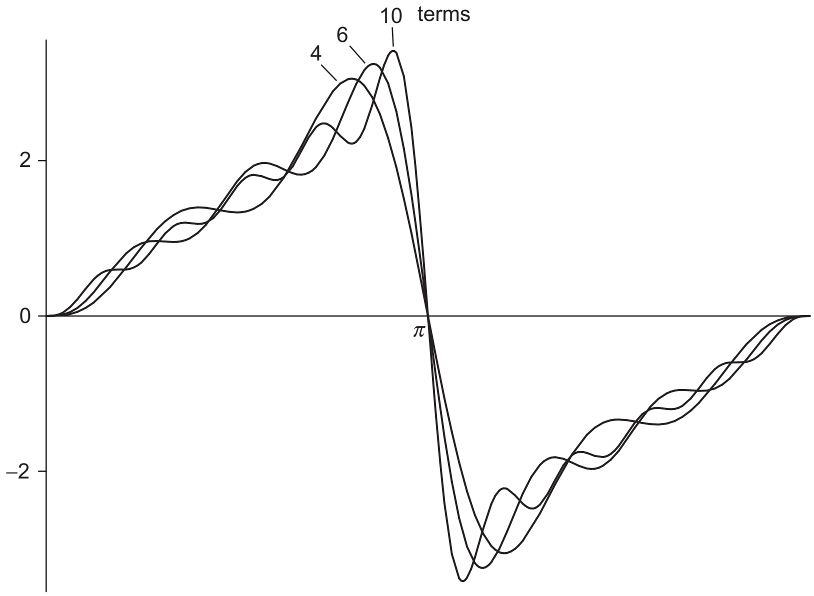

FIGURE 19.2 Expansion of sawtooth wave form, range $[0, 2\pi]$.

1. There is a steady increase in the accuracy of the representation as the number of terms included is increased.
2. At $x = \pi$, where $f(x)$ changes discontinuously from $+\pi$ to $-\pi$, all the curves pass through the average of these two values, namely $f(\pi) = 0$.
3. In the vicinity of the discontinuity at $x = \pi$, there is an overshoot that persists and shows no sign of diminishing.

As a matter of incidental interest, setting $x = \pi/2$ in Eq. (19.9) leads to

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\left[1 - 0 - \frac{1}{3} - 0 + \frac{1}{5} - 0 - \frac{1}{7} + \cdots\right],$$

thereby yielding an alternate derivation of Leibniz's formula for $\pi/4$, which was obtained by another method in Exercise 1.3.2. ■

Periodic Functions

Fourier series are used extensively to represent periodic functions, especially wave forms for signal processing. The form of the series is inherently periodic; the expansions in Eqs. (19.1) and (19.4) are periodic with period 2π, with $\sin nx$, $\cos nx$, and $\exp(inx)$, each completing n cycles of oscillation in that interval. Thus, while the coefficients in a Fourier expansion are determined from an interval of length 2π, the expansion itself (if the function involved is actually periodic) applies for an indefinite range of x. The periodicity also means that the interval used for determining the coefficients need not be $[0, 2\pi]$ but may be any other interval of that length. Often one encounters situations in which the formulas in Eqs. (19.2) and (19.3) are changed so that their integrations run between $-\pi$ and π. In fact, it would have been natural to have restated Example 19.1.1 as dealing with $f(x) = x$, for $-\pi < x < \pi$. This of course does not remove the discontinuity or change the form of the Fourier series. The discontinuity has simply been moved to the ends of the interval in x.

In actual situations, the natural interval for a Fourier expansion will be the wavelength of our wave form, so it may make sense to redefine our Fourier series so that Eq. (19.1) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{19.10}$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(s)\cos\frac{n\pi s}{L}\,ds, \qquad n = 0, 1, 2, \ldots, \tag{19.11}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(s)\sin\frac{n\pi s}{L}\,ds, \qquad n = 1, 2, \ldots. \tag{19.12}$$

In many problems the x dependence of a Fourier expansion describes the spatial dependence of a wave distribution that is moving (say, toward $+x$) with **phase velocity** v. This means that in place of x we need to write $x - vt$, and this substitution carries the implicit assumption that the wave form retains the same shape as it moves forward.[2] The individual terms of the Fourier expansion can now be given an interesting interpretation. Taking as an example the term

$$\cos\left[\frac{n\pi}{L}(x - vt)\right],$$

[2]For waves in physical media, this assumption is by no means always true, as it depends on the time-dependent response properties of the medium.

we note that it describes a contribution of wavelength $2L/n$ (when x increases this much at constant t, the argument of the cosine function increases by 2π). We also note that the period of the oscillation (the change in t at constant x for one cycle of the cosine function) is $T = 2L/nv$, corresponding to the oscillation frequency $\nu = nv/2L$. If we call the frequency for $n = 1$ the **fundamental frequency** and denote it $\nu_0 = v/2L$, we identify the terms for each $n > 1$ in the Fourier series as describing overtones, or **harmonics** of the fundamental frequency, with individual frequencies $n\nu_0$.

A typical problem for which Fourier analysis is suitable is one in which a particle undergoing oscillatory motion is subject to a periodic driving force. If the problem is described by a linear ODE, we may make a Fourier expansion of the driving force and solve for each harmonic individually. This makes the Fourier expansion a practical tool as well as a nice analytical device. We stress, however, that its utility depends crucially on the linearity of our problem; in nonlinear problems an overall solution is not a superposition of component solutions.

As suggested earlier, we have proceeded on the assumption that v, the phase velocity, is the same for all terms of the Fourier series. We now see that this assumption corresponds to the notion that the medium supporting the wave motion can respond equally well to forces at all frequencies. If, for example, the medium consists of particles too massive to respond quickly at high frequency, those components of the wave form will become attenuated and damped out of a propagating wave. Conversely, if the system contains components that resonate at certain frequencies, the response at those frequencies will be enhanced. Fourier expansions give physicists (and engineers) a powerful tool for analyzing wave forms and for designing media (e.g., circuits) that yield desired behaviors.

One question that is sometimes raised is: “Were the harmonics there all along, or were they created by our Fourier analysis?” One answer compares the functional resolution into harmonics with the resolution of a vector into rectangular components. The components may have been present, in the sense that they may be isolated and observed, but the resolution is certainly not unique. Hence many authors prefer to say that the harmonics were created by our choice of expansion. Other expansions in other sets of orthogonal functions would produce a different decomposition. For further discussion, we refer to a series of notes and letters in the *American Journal of Physics*.[3]

What if a function is not periodic? We can still obtain its Fourier expansion, but (a) the results will of course depend on how the expansion interval is chosen (both as to position and length), and (b) because no information outside the expansion interval was used in obtaining the expansion, we can have no realistic expectation that the expansion will produce there a reasonable approximation to our function.

Symmetry

Suppose we have a function $f(x)$ that is either an even or an odd function of x. If it is even, then its Fourier expansion cannot contain any odd terms (since all terms are linearly independent, no odd term can be removed by retaining others). Our expansion, developed

[3] B. L. Robinson, Concerning frequencies resulting from distortion. *Am. J. Phys.* **21**: 391 (1953); F. W. Van Name, Jr., Concerning frequencies resulting from distortion. *Am J. Phys.* **22**: 94 (1954).

for the interval $[-\pi, \pi]$, then must take the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad f(x) \text{ even.} \tag{19.13}$$

On the other hand, if $f(x)$ is odd, we must have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad f(x) \text{ odd.} \tag{19.14}$$

In both cases, when determining the coefficients we only need consider the interval $[0, \pi]$, referring to Eqs. (19.2) and (19.3), as the adjoining interval of length π will make a contribution identical to that considered. The series in Eqs. (19.13) and (19.14) are sometimes called **Fourier cosine** and **Fourier sine** series.

If we have a function defined on the interval $[0, \pi]$, we can represent it either as a Fourier sine series or as a Fourier cosine series (or, if it has no interfering singularities, as a power series), with similar results on the interval of definition. However, the results outside that interval may differ markedly because these expansions carry different assumptions as to symmetry and periodicity.

Example 19.1.2 Different Expansions of $f(x) = x$

We consider three possible ways to expand $f(x) = x$ based on its values on the range $[0, \pi]$:

- Its power-series expansion will (obviously) have the power-series expansion $f(x) = x$.
- Comparing with Example 19.1.1, its Fourier sine series will have the form given in Eq. (19.9).
- Its Fourier cosine series will have coefficients determined from

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx = \begin{cases} \pi, & n = 0, \\ -\dfrac{4}{n^2 \pi}, & n = 1, 3, 5, \ldots, \\ 0, & n = 2, 4, 6, \ldots, \end{cases}$$

corresponding to the expansion

$$f(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi} \frac{\cos(2n+1)x}{(2n+1)^2}.$$

All three of these expansions represent $f(x)$ well in the range of definition, $[0, \pi]$, but their behavior becomes strikingly different outside that range. We compare the three expansions for a range larger than $[0, \pi]$ in Fig. 19.3. ■

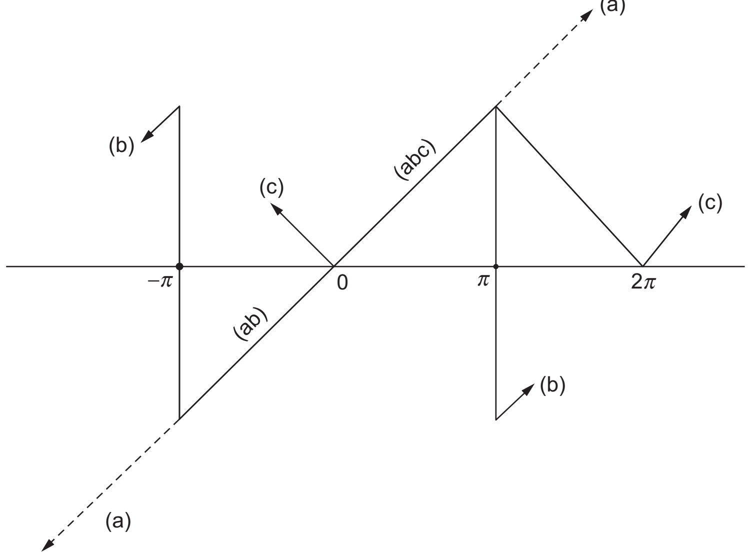

FIGURE 19.3 Expansions of $f(x) = x$ on $[0, \pi]$: (a) power series, (b) Fourier sine series, (c) Fourier cosine series.

Operations on Fourier Series

Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{19.15}$$

yields

$$\int_{x_0}^{x} f(x)\,dx = \frac{a_0 x}{2}\Big|_{x_0}^{x} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx\Big|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nx\Big|_{x_0}^{x}. \tag{19.16}$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series, Eq. (19.15), is not itself convergent. The function $f(x)$ need only be integrable. A discussion will be found in Jeffreys and Jeffreys (Additional Readings).

Strictly speaking, Eq. (19.16) may not be a Fourier series; that is, if $a_0 \neq 0$, there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x)dx - \frac{1}{2}a_0 x \tag{19.17}$$

will still be a Fourier series.

The situation regarding differentiation is quite different from that of integration. Here the word is caution. Consider the series for

$$f(x) = x, \quad -\pi < x < \pi. \tag{19.18}$$

We readily found (in Example 19.1.1) that the Fourier series is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \quad -\pi < x < \pi. \tag{19.19}$$

Differentiating term by term, we obtain

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx, \tag{19.20}$$

which is not convergent. **Warning**: Check your derivative for convergence.

For the triangular wave shown in Fig. 19.4 (and treated in Exercise 19.2.9), the Fourier expansion is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\cos nx}{n^2}, \tag{19.21}$$

which converges more rapidly than the expansion of Eq. (19.19); in fact, it exhibits uniform convergence. Differentiating term by term we get

$$f'(x) = \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\sin nx}{n}, \tag{19.22}$$

which is the Fourier expansion of a square wave,

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases} \tag{19.23}$$

Inspection of Fig. 19.3 verifies that this is indeed the derivative of our triangular wave.

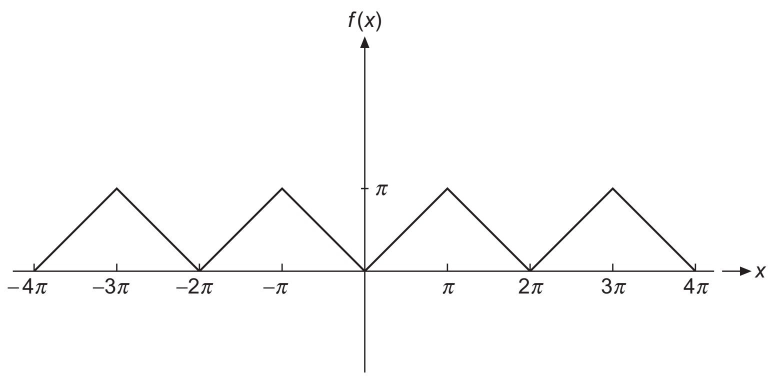

FIGURE 19.4 Triangular wave.

- As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.
- In general, term-by-term differentiation is permissible if the series to be differentiated is uniformly convergent.

Summing Fourier Series

Often the most efficient way to identify the function represented by a Fourier series is simply to identify the expansion in a table. But if it is our desire to sum the series ourselves, a useful approach is to replace the trigonometric functions by their complex exponential forms, and then identifying the Fourier series as one or more power series in $e^{\pm ix}$.

Example 19.1.3 SUMMATION OF A FOURIER SERIES

Consider the series $\sum_{n=1}^{\infty}(1/n)\cos nx$, $x \in (0, 2\pi)$. Since this series is only conditionally convergent (and diverges at $x = 0$), we take

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^n\cos nx}{n},$$

absolutely convergent for $|r| < 1$. Our procedure is to try forming power series by transforming the trigonometric functions into exponential form:

$$\sum_{n=1}^{\infty}\frac{r^n\cos nx}{n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n e^{inx}}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n e^{-inx}}{n}.$$

Now, these power series may be identified as Maclaurin expansions of $-\ln(1-z)$, with $z = re^{ix}$ or re^{-ix}. From Eq. (1.97),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{r^n\cos nx}{n} &= -\frac{1}{2}[\ln(1-re^{ix}) + \ln(1-re^{-ix})]\\ &= -\ln[(1+r^2) - 2r\cos x]^{1/2}.\end{aligned}$$

Setting $r = 1$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\cos nx}{n} &= -\ln(2-2\cos x)^{1/2}\\ &= -\ln\left(2\sin\frac{x}{2}\right), \quad (0 < x < 2\pi). \qquad (19.24)\end{aligned}$$

Both sides of this expression diverge as $x \to 0$ and as $x \to 2\pi$.[4] ■

[4]Note that the range of validity of Eq. (19.24) may be shifted to $[-\pi, \pi]$ (excluding $x = 0$) if we replace x by $|x|$ on the right-hand side.