Pole expansion of meromorphic functions

Meromorphic function: $f(z)$ analytic with only isolated poles

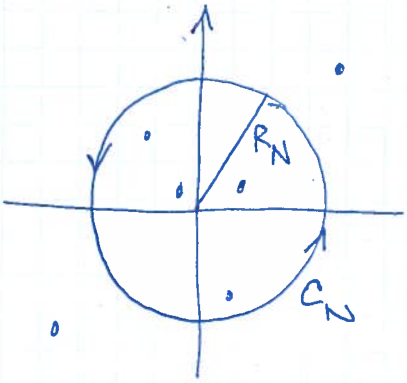
Now assume $f(z)$ meromorphic with discrete simple poles at $z_1, z_2, \dots$ with residues $b_1, b_2 \dots$ with points ordered so that $0 < |z_1| \leq |z_2| \leq \dots$. In addition $f(z)$ is analytic at $z = 0$ and $\lim_{z \to \infty} \left| \frac{f(z)}{z} \right| \to 0$ then

$$f(z) = f(0) + \sum_{n=1}^{\infty} \underbrace{b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right)}_{-\frac{z b_n}{z_n(z_n - z)}}$$

Pole expansion

Mittag-Leffler theorem

Consider the integral $I_N = \oint_{C_N} \frac{f(w)\,dw}{w(w-z)}$ where C_N is a circle enclosing the first N pole of $f(z)$



Length of C_N is $2\pi R_N$ but argument goes as $|f(R_N)|/R_N^2 \rightarrow \lim_{R_N \to \infty} I_N = 0$

C_N encircles simple poles at $w = 0$, $w = z$ and $w = z_n$, $n = 1, 2, \dots N$ and $f(w)$ is nonsingular at $w = 0$ and $w = z$ and the residue at z_n of $\frac{f(w)}{w(w-z)}$ is $\frac{b_n}{z_n(z_n - z)}$

Then $I_N = 2\pi i \frac{f(0)}{-z} + 2\pi i \frac{f(z)}{z} + 2\pi i \sum_{n=1}^{N} \frac{b_n}{z_n(z_n - z)}$

Let $N \to \infty \Rightarrow f(z) = f(0) + \sum_{n=1}^{\infty} \frac{b_n z}{z_n(z_n - z)} =$

$$= f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right)$$

Pole expansion of $\cot z$:

$\cot z = \frac{\cos z}{\sin z}$ has a simple pole at $z = 0$ with residue $= 1$

Remove the singularity by considering $\cot(z) - \frac{1}{z}$

We have simple poles at $\pm n\pi$ $(n \neq 0)$ with residues obtained using l'Hôpital's rule: wrong in the book

$$b_n = \lim_{z\to n\pi} (z-n\pi)\left(\cot z - \frac{1}{z}\right) = \lim_{z\to n\pi} (z-n\pi)\left(\frac{z\cot z - 1}{z}\right) =$$

$$= \lim_{z\to n\pi} (z-n\pi)\left(\frac{\frac{z\cos z}{\sin z} - 1}{z}\right) = \lim_{z\to n\pi} \frac{(z-n\pi)(z\cos z - \sin z)}{z\sin z} =$$

$$= \left.\frac{\frac{d}{dz}\{(z-n\pi)(z\cos z - \sin z)\}}{\frac{d}{dz} z\sin z}\right|_{z=n\pi} = \left.\frac{z\cos(z) - \sin(z) + (z-n\pi)(-z\sin z)}{\sin z + z\cos z}\right|_{z=n\pi}$$

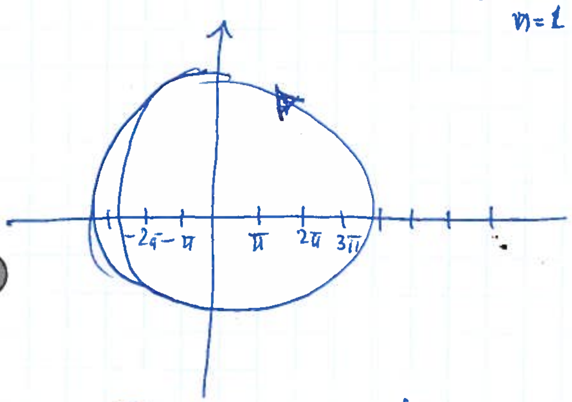
$$= 1 \Rightarrow b_n = 1$$

Now $\cot z = \dfrac{\cos(z)}{\sin(z)} = \dfrac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}} = \dfrac{1}{z}\left\{\dfrac{1 - z^2/2 + z^4/4! - \dots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}\right\} \xrightarrow[z\to 0]{} \dfrac{1}{z}$

so that $\lim_{z\to 0} \cot(z) - \frac{1}{z} = 0$

Thus, $\cot z - \dfrac{1}{z} = \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi} + \frac{1}{z+n\pi} + \frac{1}{-n\pi}\right) =$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2-(n\pi)^2}$$



$$\Rightarrow \cot(z) = \frac{1}{z} + 2z\left\{\frac{1}{z^2-\pi^2} + \frac{1}{z^2-(2\pi)^2} + \frac{1}{z^2-(3\pi)^2} + \dots\right\}$$

Counting poles and zeros:

Consider the logarithmic derivative $\frac{d}{dz}\ln f(z) = \frac{f'(z)}{f(z)}$

Assume that $f(z)$ has a zero of order μ at $z=z_0$. Then we can write $f(z) = (z-z_0)^{\mu} g(z)$ with $g(z)$ finite and non-zero at z_0

This gives $f'(z) = \mu(z-z_0)^{\mu-1} g(z) + (z-z_0)^{\mu} g'(z)$

and $\dfrac{f'(z)}{f(z)} = \dfrac{\mu(z-z_0)^{\mu-1}g(z) + (z-z_0)^{\mu}g'(z)}{(z-z_0)^{\mu}g(z)} = \dfrac{\mu}{z-z_0} + \dfrac{g'(z)}{g(z)}$ (*)

Consider a pole of order m at $z=z_0$

$f(z) = (z-z_0)^{-m} g(z)$ with $g(z)$ as before

We find $f'(z) = -m\,(z-z_0)^{-m-1} g(z) + (z-z_0)^{-m} g'(z)$

and $$\frac{f'(z)}{f(z)} = -\frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

so that $$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \left[\sum_{i=1}^{N} \mu_i \{\text{enclosed zeros}\} - \sum_{j=1}^{M} \mu_j \{\text{enclosed poles}\}\right]$$

Product expansion of entire functions:

A function $f(z)$ that is analytic for all finite z is called an entire function. If $f(z)$ is an entire function then $f'(z)/f(z)$ will be meromorphic with all its poles simple. Assume that also the zeros of f are simple and at points z_n so that μ in $(*)$ is 1. Do a pole expansion of $f'(z)/f(z)$:

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{z-z_n} + \frac{1}{z_n}\right]$$

Integrate $$\int_0^z \frac{f'(z)}{f(z)}\,dz = \ln f(z) - \ln f(0) =$$

$$= \frac{z f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\ln(z-z_n) - \ln(-z_n) + \frac{z}{z_n}\right]$$

Exponentiate: $$f(z) = f(0)\exp\left\{\frac{z f'(0)}{f(0)}\right\}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{z/z_n}$$

$\cos(z)$: $\quad z_n = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi \quad , \quad -\infty < n < +\infty$

$f'(0) = \sin(0) = 0$

$$\cos(z) = \prod_{n=-\infty}^{\infty}\left(1-\frac{z}{(n+\frac{1}{2})\pi}\right)e^{z/(n-\frac{1}{2})\pi} = \prod_{n=1}^{\infty}\left(1-\frac{z}{(n-\frac{1}{2})\pi}\right)e^{\frac{z}{(n-\frac{1}{2})\pi}}\cdot\left(1+\frac{z}{(n-\frac{1}{2})\pi}\right)e^{-\frac{z}{(n-\frac{1}{2})\pi}}$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right)$$

For $\sin(z)$ apply to $\frac{\sin(z)}{z}$ such that $f(0) = 1$

$$\Rightarrow \quad \sin(z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

Evaluation of sums:

The cotangent is a meromorphic function with regularly spaced poles at $\pm n\pi$ so that $\cot \pi z$ has poles at all integers on the real axis with residue

$$\left.\frac{\cos \pi z}{\frac{d}{dz}\sin \pi z}\right|_{z=n} = \frac{1}{\pi}$$

so that $\pi \cot \pi z$ has residue 1 for all integers.

Assume the function $f(z)$ only has isolated singularities at points z_j that are not real integers and that

- $z f(z) \xrightarrow[|z|\to\infty]{} 0$
- The integral $\oint_{C_N} f(z)\pi \cot \pi z \, dz = I_N$ with C_N a circle of radius $N+\frac{1}{2}$ centered at $z=0$ becomes

$$I_N = 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum_j \{\text{residues of } f(z)\pi\cot\pi z \text{ at } z_j\}$$

But $z f(z) \to 0$ and $\frac{\cos z}{\sin z} = i\,\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$ of order 1 for large $|z|$ if not very close to a pole

- so $I_N \longrightarrow 0$
- and $\sum_{n=-\infty}^{\infty} f(n) = -\sum_j \{\text{residues of } f(z)\pi\cot\pi z \text{ at } z_j\}$

The sum $S = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ is symmetric in $\pm n$ so

$$S = \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} \quad ; \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} - \underbrace{\frac{1}{a^2}}_{n=0} = 2S$$

$$f(z) = \frac{1}{z^2+a^2} \quad ; \quad z f(z) \xrightarrow[|z|\to\infty]{} 0$$

with simple poles at $z = \pm ia$. We need the residues

$$\lim_{z\to ia} \frac{(z-ia)\pi\cot\pi z}{(z-ia)(z+ia)} = \frac{\pi \cot i\pi a}{2ia} = -\frac{\pi \coth \pi a}{2a}$$

$$\lim_{z\to -ia} \frac{(z+ia)\pi\cot\pi z}{(z+ia)(z-ia)} = \frac{\pi\cot(-i\pi a)}{-2ia} = \frac{-\pi\coth(-\pi a)}{-2a} = -\frac{\pi\coth(\pi a)}{2a} \quad 5$$

$$\Rightarrow \quad 2S = \frac{\pi\cot\pi a}{a} - \frac{1}{a^2} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi\coth\pi a}{2a} - \frac{1}{2a^2}$$

Similarly $\pi\tan\pi z$ has poles at $n+\frac{1}{2}$ with residue $+1$

$\frac{\pi}{\sin\pi z} = \pi\csc\pi z$ — " — integer n — " — ± 1

$\frac{\pi}{\cos\pi z} = \pi\sec\pi z$ — " — half integer n — " ———

Orthogonal Polynomials

- Rodrigues formulas
- Schlaefli integral
- Generating functions

General second-order Sturm-Liouville ODE

$$p(x)y'' + q(x)y' + \lambda y = 0$$

with $p(x) = \alpha x^2 + \beta x + \gamma$, $q(x) = \mu x + \nu$

For a polynomial solution to the ODE we have $y_n(x) = \sum_{j=0}^{n} g_j x^j$

Consider the power x^n in the ODE. The coefficient becomes

$$n(n-1)\alpha g_n + n\mu g_n + \lambda_n g_n = 0 \quad \Rightarrow \lambda_n = -n(n-1)\alpha - n\mu$$

The ODE can be made self-adjoint by factor

$$w(x) = \frac{1}{p} \exp\left\{ \int^x \frac{q(x)dx}{p(x)} \right\} \quad \text{with} \quad (wp)' = wq \Leftrightarrow w' = w\frac{q-p'}{p}$$

Solutions are now given by

$$\boxed{y_n(x) = \frac{1}{w(x)} \left(\frac{d}{dx}\right)^n \left[w p(x)^n\right] \quad \text{Rodrigues formula}}$$

Proof:

Consider $p[wp^n]' = p\{w'p^n + nwp'p^{n-1}\} =$

$$= p\left\{ w\frac{(q-p')}{p}p^n + nwp'p^{n-1} \right\} = w\left\{(q-p')p^n + npp'p^n\right\} =$$

$$= wp^n\{(n-1)p' + q\}$$

p is quadratic in x and q is linear

Differentiate $n+1$ times and divide by w (we want to construct y_n)

Leibnitz' formula: $\frac{d^n}{dx^n}\left[A(x)B(x)\right] = \sum_{s=0}^{n} \binom{n}{s} \frac{d^{n-s}}{dx^{n-s}} A(x) \frac{d^s}{dx^s} B(x)$

where $\binom{n}{s} = \frac{n!}{s!(n-s)!}$

$$\frac{p}{w}\frac{d^{n+2}}{dx^{n+2}}\left[wp^n\right] + \frac{(n+1)p'}{w}\frac{d^{n+1}}{dx^{n+1}}\left[wp^n\right] + \frac{n(n+1)p''}{2w}\frac{d^n}{dx^n}\left[wp^n\right] =$$

$$= \frac{(n+1)p'+q}{w}\frac{d^{n+1}}{dx^{n+1}}\left[wp^n\right] + \frac{(n+1)\left[(n-1)p''+q'\right]}{w}\left(\frac{d}{dx}\right)^n\left[wp^n\right]$$

Only three terms on the left and two on the right since $p(x)$ is second order and $q(x)$ linear

Move over and collect terms

$$\frac{p}{w}\left(\frac{d}{dx}\right)^{n+2}\left[wp^n\right] + \frac{2p'-q}{w}\left(\frac{d}{dx}\right)^{n+1}\left[wp^n\right] + \underbrace{\left\{\frac{n(n+1)p''}{2w} - \frac{(n+1)\left[(n-1)p''+q'\right]}{w}\right\}}_{-\left[\frac{n^2-n-2}{2}p'' + (n+1)q'\right]\frac{1}{w}}\left(\frac{d}{dx}\right)^n\left[wp^n\right] = 0$$

The last term reduces to a factor times y_n

We want to construct derivatives of $\frac{1}{w}\left(\frac{d}{dx}\right)^n\left[wp^n\right] = y_n$

Take the term $\left(\frac{d}{dx}\right)^{n+2}$: $\frac{p}{w}\left(\frac{d}{dx}\right)^{n+2}\left[wp^n\right]$;

$$\underbrace{\frac{d^2}{dx^2}\left\{\frac{1}{w}\left(\frac{d}{dx}\right)^{n\cancel{+2}}\left[wp^n\right]\right\}}_{y_n''} = \underline{\underline{\frac{1}{w}\left(\frac{d}{dx}\right)^{n+2}\left[wp^n\right]}} + 2\frac{dw^{-1}}{dx}\left(\frac{d}{dx}\right)^{n+1}\left[wp^n\right] + \frac{d^2w^{-1}}{dx^2}\left(\frac{d}{dx}\right)^n\left[wp^n\right]$$

$$\frac{d}{dx}\frac{1}{w} = -\frac{w'}{w^2} = -\frac{1}{w}\cdot\frac{q-p'}{p}$$

$$\frac{d^2}{dx^2}\left(\frac{1}{w}\right) = -\frac{q'-p''}{pw} + \frac{(q-p')p'}{p^2w} + \frac{q-p'}{p}\frac{w'}{w^2} = -\frac{q'-p''}{pw} + \frac{q(q-p')}{p^2w}$$

which gives $\frac{p}{w}\left(\frac{d}{dx}\right)^{n+2}\left[wp^n\right] = py_n'' + 2\frac{(q-p')}{w}\left(\frac{d}{dx}\right)^{n+1}\left[wp^n\right] -$

$$-\left[p''-q' + \frac{q(q-p')}{p}\right]y_n$$

Insert in the equation

$$py_n'' + \left[\frac{2(q-p')}{w} + \frac{2p'-q}{w}\right]\left(\frac{d}{dx}\right)^{n+1}\left[wp^n\right] - \left[\frac{n^2-n-2}{2}p'' + (n+1)q' +\right.$$

$$\left. + p''-q' + \frac{q(q-p')}{p}\right]y_n =$$

$$\Rightarrow p y_n'' + \frac{q}{w}\left(\frac{d}{dx}\right)^{n+1}[wp^n] - \left[\frac{n^2-n}{2}p'' + nq' + \frac{q(q-p')}{p}\right] y_n = 0$$

Use $\displaystyle \frac{q}{w}\left(\frac{d}{dx}\right)^{n+1}[wp^n] = q\frac{d}{dx}\underbrace{\left[\frac{1}{w}\left(\frac{d}{dx}\right)^n[wp^n]\right]}_{y_n'} + \frac{q(q-p')}{p} y_n$

from $\displaystyle \frac{dw^{-1}}{w}\cdot\left(\frac{d}{dx}\right)^n[wp^n]$

$$\Rightarrow p y_n'' + q y_n' + \underbrace{\left\{\frac{q(q-p')}{p} - \frac{n^2-n}{2}p'' - nq' - \frac{q(q-p')}{p}\right\}}_{-\left[\frac{n^2-n}{2}p'' - nq'\right] y_n} y_n = 0$$

but $p'' = 2\alpha$ and $q' = \mu$

so that we have recovered the eigenvalue

$\lambda_n = -n(n-1)\alpha - n\mu$ and

$\displaystyle y_n(x) = \frac{1}{w(x)}\left(\frac{d}{dx}\right)^n[w(x)p(x)^n]$ solves the equation.

Hermite ODE $\quad y'' - 2xy' + \lambda y = 0 \qquad p(x) = 1, \; q(x) = -2x$

$$w(x) = \exp\left\{\int^x (-2x)dx\right\} = e^{-x^2}$$

$$\Rightarrow H_n(x) = \frac{(-1)^n}{w}\left(\frac{d}{dx}\right)^n[wp^n] = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

convention

Schlaefli integral

Use Cauchy's integral formula to do the multiple differentiations in Rodrigues formula

$$y_n(x) = \frac{1}{w(x)}\,\frac{n!}{2\pi i}\oint_C \frac{w(z)[p(z)]^n}{(z-x)^{n+1}}dz$$

with $w(z)[p(z)]^n$ analytic everywhere on and inside C

Generating functions

Let $f_n(x)$ be a set of functions defined for integer values n there are cases when $f_n(x)$ can be described as the coefficients of the powers of an auxiliary variable t in the expansion of a function $g(x,t)$: the generating fctn

$$g(x,t) = \sum_n c_n f_n(x) t^n$$

with $n \geq 0$ we have a Taylor expansion

$-\infty < n < \infty$ Laurent expansion

We obtain $c_n f_n(x) = \frac{1}{2\pi i} \oint_C \frac{g(x,t)}{t^{n+1}} dt$ with C encircling $t=0$ but no other singularities ~~wrt~~ with respect to t

Taking derivatives of the generating function we can derive recurrence relations and the ODE that $f_n(x)$ satisfy, e.g.

$$\frac{\partial g(x,t)}{\partial t} = \sum_n n c_n f_n(x) t^{n-1} = \sum_n (n+1) c_{n+1} f_{n+1}(x) t^n$$

Example Hermite polynomials

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$\frac{\partial}{\partial t} e^{-t^2+2tx} = (2x-2t) e^{-t^2+2tx} = \sum_n n H_n(x) \frac{t^{n-1}}{n!}$$

$$\sum_{n=0}^{\infty} 2x H_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} 2 H_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} n H_n \frac{t^{n-1}}{n!}$$

$$t^n: \quad 2x \frac{H_n}{n!} - \frac{2 H_{n-1}}{(n-1)!} = \frac{(n+1) H_{n+1}}{(n+1)!}$$

$$\Rightarrow 2x H_n(x) - 2n H_{n-1}(x) = H_{n+1}(x)$$

Take derivative with respect to x

$$\frac{\partial}{\partial x} g(x,t) = 2te^{-t^2+2tx} = \sum_{n=0}^{\infty} \mathcal{H}_n'(x)\frac{t^n}{n!}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} 2\,\mathcal{H}_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} \mathcal{H}_n'(x)\frac{t^n}{n!}$$

$$\Rightarrow \quad \mathcal{H}_n'(x) = 2n\,\mathcal{H}_{n-1}(x)$$