Complex Algebra

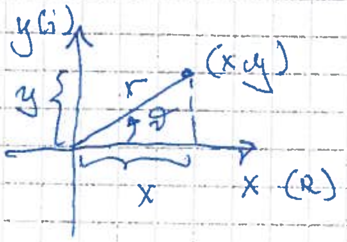
The complex numbers are ordered pairs $z=(x,y)$ where x and y are real and $z=x+iy$ with $i^2=-1$

Addition of complex numbers is done for each component separately $z_1+z_2=(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$

Multiplication is defined as

$$z_1\cdot z_2=(x_1+iy_1)(x_2+iy_2)=x_1x_2+ix_1y_2+iy_1x_2-y_1y_2=$$
$$=(x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)$$

We can visualize z in the complex plane (2-dim)



and we see that $\begin{cases} x=r\cos\vartheta \\ y=r\sin\vartheta \end{cases}$

i.e. $z=r(\cos\vartheta+i\sin\vartheta)=r(\cos\vartheta,\sin\vartheta)$

All elementary fctns can be defined on the complex plane, e.g. $\sin x\to\sin z$, $e^x\to e^z$

The expansion ~~of~~ in powers of z: $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$

With $z=i\vartheta$ we have $e^{i\vartheta}=\sum_{n=0}^{\infty}\frac{(i\vartheta)^n}{n!}=\underbrace{\sum_{\nu=0}^{\infty}\frac{(i\vartheta)^{2\nu}}{(2\nu)!}}_{\text{even}}+\underbrace{\sum_{\nu=0}^{\infty}\frac{(i\vartheta)^{2\nu+1}}{(2\nu+1)!}}_{\text{odd}}$

$$=\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\vartheta^{2\nu}}{(2\nu)!}+i\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{\vartheta^{2\nu+1}}{(2\nu+1)!}=\cos\vartheta+i\sin\vartheta$$

$\therefore$ We have an alternative representation as

$$z=re^{i\vartheta}$$

$r=|z|=\sqrt{x^2+y^2}$

$\vartheta=\operatorname{atan}\left(\frac{x}{y}\right)$

special cases $e^{i\pi/2}=i$

$e^{i\pi}=-1$

$e^{i(\varphi+2\pi)}=e^{i\varphi}$ periodic

Addition & subtraction most convenient with the vector (pairs) representation

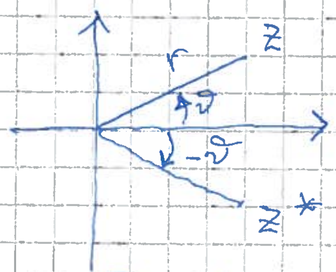
The exponential form useful for multiplication, division powers etc $\frac{1}{z} = \frac{1}{re^{i\vartheta}} = \frac{1}{r} e^{-i\vartheta}$ $\quad z^{1/3} = (re^{i\vartheta})^{1/3} = r^{1/3} e^{i\vartheta/3}$

Complex valued functions $w(z)$ can be ~~divided~~ separated into a real and an imaginary part $w(z) = u(x,y) + i v(x,y)$. where both u and v are real functions

Ex. $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$

The function is a mapping transforming the pair (x,y) into the pair $(u(x,y), v(x,y))$.

Complex conjugation corresponds to a reflection through the real axis $\quad z^* = x - iy = (x+iy)^* = (re^{i\vartheta})^* = re^{-i\vartheta}$



De Moivre formula: Taking the nth power of $e^{i\vartheta}$ gives

$$(e^{i\vartheta})^n = e^{in\vartheta} = (\cos\vartheta + i\sin\vartheta)^n$$

but $e^{in\vartheta} = \cos(n\vartheta) + i\sin(n\vartheta)$ so that

$$\cos(n\vartheta) + i\sin(n\vartheta) = (\cos\vartheta + i\sin\vartheta)^n$$

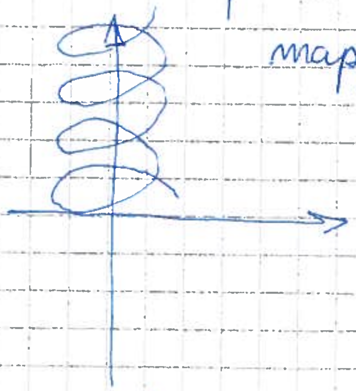
Binomial expansion of the right hand side and comparison of real and imaginary parts separately gives

$$\cos(n\vartheta) + i\sin(n\vartheta) = \sum_{k=0}^{n} \binom{n}{k} (\cos\vartheta)^{n-k} (i\sin\vartheta)^k$$

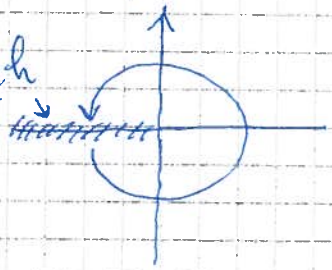
$$\Rightarrow \cos(n\vartheta) = \sum_{j=0}^{[n/2]} \binom{n}{2j} (\cos\vartheta)^{n-2j} (-1)^j (\sin\vartheta)^{2j}$$

and $\sin(n\vartheta) = \sum_{j=0}^{[(n-1)/2]} \binom{n}{2j+1} (-1)^j (\cos\vartheta)^{n-(2j+1)} (\sin\vartheta)^{2j+1}$

With the logarithm we encounter the first example of a multivalued function when it is taken into the complex plan[e]. We have that $\ln z = \ln(x+iy)$ or rather $\ln z = \ln(re^{i\vartheta})$ $= \ln r + i\vartheta$, but $z = re^{i\vartheta} = re^{i(\vartheta + 2n\pi)}$ results in $\ln z = \ln r + i(\vartheta + 2n\pi)$ so the value of the function depends on the phase $2n\pi$ which added to z still represents the point z, that is each point z is mapped onto an infinite number of points by $\ln z$



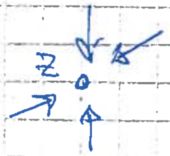
To ensure a unique mapping we have to introduce a cut of the complex plane (cut line or branch cut), i.e. limit the phase to e.g. $(-\pi, \pi]$



We have defined complex valued functions $w(z) = u(x,y) + i\,v(x,y)$ and now wish to define differentiation and integration with respect to these functions.

The derivative is defined in the usual way as

$$f'(z) = \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{z + \delta z - z}$$



We can now arrive at z from different directions since $\delta z = (\delta x, \delta y)$

For real valued functions the derivative exists if left and right derivatives are equal i.e. $\lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{x+\delta x - x} = \lim_{\delta x\to 0} \frac{f(x)-f(x-\delta x)}{x-x+\delta x}$

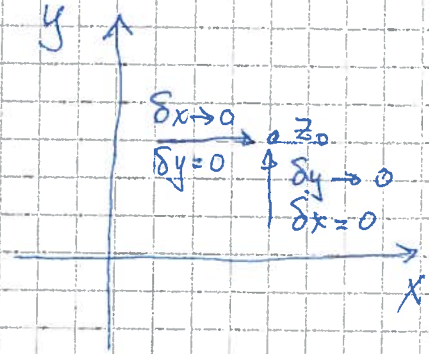
Similarly, for the derivative of a complex valued function to exist ~~as~~ the derivative has to be independent of the direction $\delta z = (\delta x, \delta y) = \delta x + i\delta y$. This results in the powerful Cauchy-Riemann conditions

The change in the function is given by

$$\delta f = \delta u + i\delta v$$

and $$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

Consider the x and y directions separately



$$\delta x: \lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\delta y: \lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta y\to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For the derivative to exist these two expressions must be equal, i.e. $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$$\boxed{\begin{array}{ll} \text{Real part:} & \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \text{Imaginary part:} & \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}}$$

Cauchy-Riemann conditions

Potential theory

The tangents of $u(x,y)$ and $v(x,y)$ are given by $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$. These are orthogonal to each other since the scalar product $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} = 0$$ at any common point

Conversely: If the Cauchy-Riemann conditions are fulfilled and the partial derivatives are continuous then the function is differentiable

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$ (derivatives continuous,

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}}$$

Cauchy-Riemann: $\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$

$$\Rightarrow \frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}} =$$

$$= \frac{\frac{\partial u}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right) + i\frac{\partial v}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right)}{1 + i\frac{\delta y}{\delta x}} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

independent of direction as long as the derivatives are continuous.

Analytical functions

If $f(z)$ is differentiable at z_0 and in a small neighborhood of z_0 then $f(z)$ is analytical at z_0. If $f'(z)$ is undefined for $z = z_0$ then z_0 is a singular point

Ex. $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u(x,y)} + i\underbrace{2xy}_{v(x,y)}$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Derivatives continuous and Cauchy-Riemann satisfied
$\Rightarrow$ f' exists everywhere and $f(z)$ is analytical

Ex. $f(z) = z^* = \underbrace{x}_{u} \underbrace{- iy}_{v} \Rightarrow \frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$

Analyticity gives more information on the function than regular differentiation ~~of~~ for real functions

1) Cauchy-Riemann gives connection between the derivatives of the real and imaginary parts

2) Both $u(x,y)$ and $v(x,y)$ in the analytical function $f(z) = u(x,y) + iv(x,y)$ solve the Laplace equation since

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$

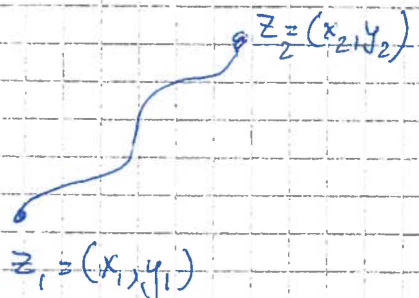
$\therefore$ Much more than just local information

Integration

The line integral from z_1 to z_2 is given by

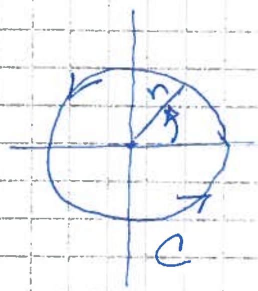
$$\int_{z_1}^{z_2} f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} \big(u(x,y)+iv(x,y)\big)(dx+i\,dy) = \int_{(x_1,y_1)}^{(x_2,y_2)} \{u(x,y)\,dx - v(x,y)\,dy\} +$$

$$+ \; i \int_{(x_1,y_1)}^{(x_2,y_2)} \{u(x,y)\,dy + v(x,y)\,dx\}$$



with the integral taken along a specific line connecting z_1 and z_2

Important special case: $\int_C z^n dz$ with C a closed circle with radius $r>0$



Take $z = re^{i\vartheta}$ $\quad dz = ire^{i\vartheta}d\vartheta$

Consider the integral $\dfrac{1}{2\pi i}\displaystyle\int_C z^n dz = \dfrac{1}{2\pi i}\int_0^{2\pi} r^n e^{in\vartheta} \cdot ire^{i\vartheta} d\vartheta =$

$$= \frac{r^{n+1}}{2\pi}\int_0^{2\pi} e^{i(n+1)\vartheta} d\vartheta = \frac{r^{n+1}}{2\pi i(n+1)}\Big[e^{i(n+1)\vartheta}\Big]_0^{2\pi} = 0 \quad \text{if } n \neq -1$$

For $n=-1$ we obtain $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ire^{i\vartheta}d\vartheta}{re^{i\vartheta}} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta = 1$

Note that we are back at the same point but the value of the logarithm $\ln z$ has increased by 2π. When integrating complex valued functions the value is that obtained along the continuous path.

Note also that the only contribution is obtained from $\frac{1}{z}$ which diverges $(\to \infty)$ when $|z| \to 0$

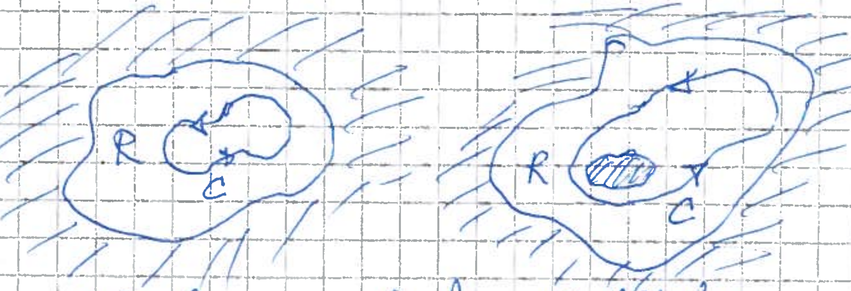
i.e. $z=0$ is a singular point (pole) for $\frac{1}{z}$

Cauchy integral theorem.

If $f(z)$ is analytical in a simply connected region R and $f(z)$ is ~~single-valued~~ (not necessary) then the integral over every closed path C in R vanishes.

Since $f(z)$ is analytical its partial derivatives are continuous.

$$\int_C f(z)dz = \oint_C f(z)dz = 0$$



Simply connected   Multiply connected

Use Stokes' theorem

$$\oint \underline{V} d\underline{\lambda} = \int_S \nabla \times \underline{V} \cdot d\underline{\sigma}$$

valid if $\underline{V}$ has continuous derivatives

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \underbrace{\oint_C (udx-vdy)}_{(1)\ \text{real}} + i\underbrace{\oint_C (vdx+udy)}_{(2)\ \text{real}}$$

In Stokes' theorem we have $\underline{V} = V_x \underline{x} + V_y \underline{y}$ and

$$\oint (V_x dx + V_y dy) = \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx dy$$

Identify in (1) $u \equiv V_x$ and $-v \equiv V_y$ which gives

$$\oint (udx - vdy) = \oint (V_x dx + V_y dy) = \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx dy =$$

$$= \int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx dy$$

In (2) we have $v \equiv V_x$ and $u \equiv V_y$ which gives

$$\oint (vdx + udy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx dy$$

but since $f(z)$ is analytical the Cauchy-Riemann conditions apply

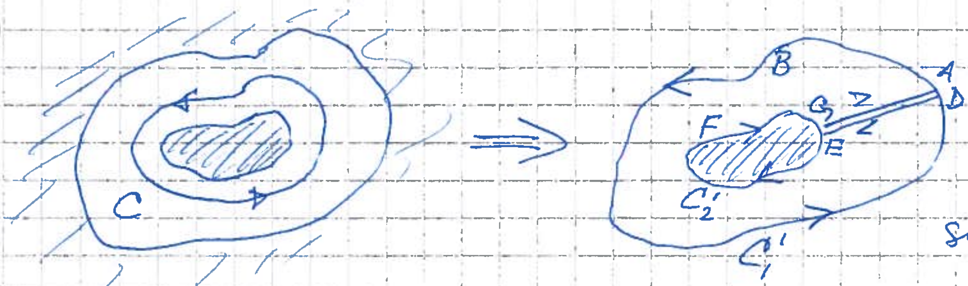
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and both integrals become $= 0$.

i.e. $\oint_C f(z)dz = 0$ with the assumption of continuous derivatives in R (not required, but necessary for this proof)

Two Comments:

1) The requirement of continuous derivatives (necessary for Stokes' theorem to be applicable) is not strict, see Cauchy-Goursot proof (not required). Specifically, the value of the integral only depends on the end points (conservative force field) $\int_{z_1}^{z_2} f(z)\,dz = F(z_1) - F(z_2) = -\int_{z_2}^{z_1} f(z)\,dz$

2) A non-simply connected region can be made simply connected (not strictly necessary for an analytical function on the region which anyway has to be single-valued and continuous).



We have $\oint_{C'} f(z)\,dz =$

$= \int_{ABD} f(z)\,dz + \int_{EFG} f(z)\,dz = 0$

since $\int_G^A f(z)\,dz = -\int_E^D f(z)\,dz$

(Cut is infinitely thin)

with $EFG = -C_2'$ we obtain

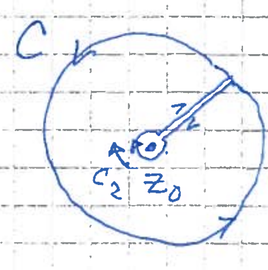
$$\oint_{C_1'} f(z)\,dz = \oint_{C_2'} f(z)\,dz$$

independent of the path

Cauchy integral formula

Let $f(z)$ be analytical on the closed contour C and in the region contained by C. Then $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$ where z_0 is an arbitrary point contained within C

z_0 is not on the contour $\Rightarrow z \neq z_0$ and the integrand is well behaved.



$f(z)$ is analytical but $\frac{f(z)}{z-z_0}$ analytical at $z=z_0$ only if $f(z_0)=0$

If the contour is deformed to introduce a cutline we can apply the Cauchy integral theorem and $\oint_C \frac{f(z)}{z-z_0}\,dz - \oint_{C_2} \frac{f(z)}{z-z_0}\,dz = 0$ (contributions along cut line cancel)

We have introduced an explicit minus sign so that the integral runs counterclockwise also for C_2. For the integral over C_2 we take the circle $z = z_0 + re^{i\vartheta}$. r is small and we'll take the limit $r \to$

We have
$$\oint_{C_2} \frac{f(z)}{z-z_0}\,dz = \oint_{C_2}^{2\pi} \frac{f(z_0+re^{i\vartheta})}{re^{i\vartheta}} \underbrace{rie^{i\vartheta}d\vartheta}_{dz} = i\oint_{C_2}^{2\pi} f(z_0+re^{i\vartheta})\,d\vartheta \equiv I$$

$$\lim_{r\to 0} I = i\oint_{C_2}^{2\pi} f(z_0)\,d\vartheta = if(z_0)\int_0^{2\pi} d\vartheta = 2\pi i f(z_0)$$ (f continuous at $z=z_0$ due to analyticity)

$\therefore$ The value ~~at~~ $f(z_0)$ is determined by the value of $f(z)$ on the contour encircling $z=z_0$. The contour is furthermore arbitrary!

Derivatives on integral form + analytical function has continuous derivatives to all orders.

We now have an expression for $f(z_0)$ and can thus construct
$$\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{1}{2\pi i\,\delta z_0}\left\{\oint \frac{f(z)}{z-z_0-\delta z_0}\,dz - \oint \frac{f(z)}{z-z_0}\,dz\right\}$$

Take the limit $\delta z_0 \to 0$: $f'(z_0) = \lim_{\delta z_0\to 0} \frac{1}{2\pi i\,\delta z_0}\left\{\oint \frac{(z-z_0)f(z)-(z-z_0-\delta z_0)f(z)}{(z-z_0-\delta z_0)(z-z_0)}\right\}$

$$= \lim_{\delta z_0\to 0} \frac{1}{2\pi i\,\delta z_0}\cdot \delta z_0 \oint \frac{f(z)}{(z-z_0-\delta z_0)(z-z_0)}\,dz = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}\,dz$$

Similarly $f^{(2)}(z_0) = \lim_{\delta z_0\to 0} \frac{f^{(1)}(z_0+\delta z_0)-f^{(1)}(z_0)}{\delta z_0} = \frac{2}{2\pi i}\oint \frac{f(z)}{(z-z_0)^3}\,dz$

General: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$, i.e. if $f(z)$ is analytica[l] then it has derivatives to all orders and the derivatives are also analytical since δz arbitrary

Note:

Compare the Taylor expansion
$$f(z_0) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$ and remember that $$\oint z^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Morera's theorem – reverse of Cauchy integral theorem

If a function $f(z)$ is continuous in a simply connected region R and $\oint f(z)dz = 0$ for every closed contour C in R then $f(z)$ is analytical in R.

Since $\oint f(z)dz = 0$ the integral of $f(z)$ in R depends only on the start and end points, that is $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$ for some function F. Consider $\frac{F(z_2)-F(z_1)}{z_2 - z_1} = \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} f(z)dz$

Since $f(z)$ is continuous we have $\lim_{z_1 \to z_2} \frac{1}{z_2 - z_1}\int_{z_1}^{z_2} f(z)dz = f(z_2)$

From the definition of derivative $\lim_{z_1 \to z_2} \frac{F(z_2)-F(z_1)}{z_2 - z_1} = F'(z)\big|_{z=z_2} = f(z_2)$

which exists since $\oint f(z)dz = 0$, i.e. the value $\int_{z_1}^{z_2} f(z)dz$ is independent of the path. Since $f(z)$ is the derivative of an analytical function then $f(z)$ is also analytical. z_2 arbitrary in R

Note: The mean value theorem does not apply to complex functions

Cauchy inequality: Let $f(z) = \sum_n a_n z^n$ be analytical and bounded such that $|f(z)| \leq M$ on a circle of radius r around the origin. Then $|a_n| r^n \leq M$ and we have an upper bound to the coefficients in the Taylor expansion.

Proof: Let $M(r) = \max_{|z|=r} |f(z)|$

Use the Cauchy integral for a_n

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \leq M(r) \frac{2\pi r}{2\pi r^{n+1}} = \frac{M(r)}{r^n}$$

where the inequality follows from

Exercise 6.3.2 $\left| \int_C f(z)dz \right| \leq |f|_{max} \cdot L$ where $|f|_{max}$ is the maximum value of $|f(z)|$ along the contour C of length L

The Liouville theorem follows directly.

If $f(z)$ is analytical and bounded in the full complex plane then $f(z) = C$ (a constant)

We have that $|a_n| \leq M r^{-n}$. Let $r \to \infty \Rightarrow |a_n| \to 0$ and $f(z) = a_0$, i.e. for the function not to be a constant there has to be at least one singularity

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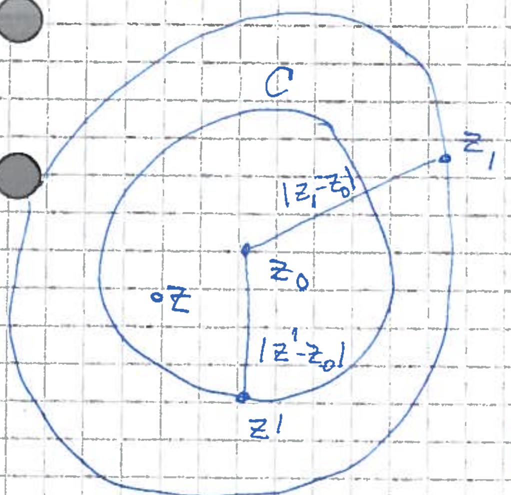
Taylor expansion:



We want to expand $f(z)$ around the point z_0 where $f(z)$ is analytical. Let z_1 be the closest point at which $f(z)$ is not analytical. On and inside the circle C with radius less than $|z_1 - z_0|$ $f(z)$ is analytical (z_1 is the closest point where $f(z)$ is not analytical)

We can use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz' = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z' - z_0) - (z - z_0)} =$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z' - z_0)\left[1 - \frac{z - z_0}{z' - z_0}\right]}$$ with z' on C and z inside C

We have not yet derived the binomial theorem for complex numbers so instead we can use that

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n$$ which is convergent for $|t| <$.

Since $|z - z_0| < |z' - z_0|$ we have

$$\frac{1}{1 - \frac{z - z_0}{z' - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$ and

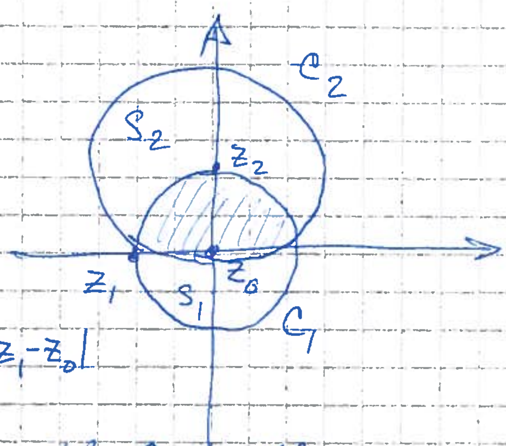
$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')\,dz'}{(z'-z_0)^{n+1}}$$

Since the series is uniformly convergent we can change the order of integration and summation:

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

From this we can derive the binomial theorem as the Taylor expansion: $(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2} z^2 + \ldots = \sum_{n=0}^{\infty} \binom{m}{n} z^n \qquad (6.5.2)$

Analytic Continuation



Let $f(z)$ be defined and analytical within the circle C_1 with z_1 as the nearest singularity. $f(z)$ can e.g. be given by its Taylor expansion with radius of convergence $r < |z_1 - z_0|$

Expand around a different point z_2 inside C_1 with $|z_2 - z_1| > |z_1 - z_0|$. Within the overlapping region $f(z)$ is uniquely defined. Within the part of S_2 outside S_1 $f(z)$ is uniquely defined by its Taylor expansion around z_2 even if the expansion around z_0 is not convergent in that region. $f(z)$ is furthermore analytical in S_2 and we have an analytic continuation of the definition of $f(z)$.

Ex. $f(z) = \frac{1}{1+z}$ has a pole $(1+z \to 0)$ at $z = -1$ but is analytical everywhere else

The expansion $\frac{1}{1+z} = 1 - z + z^2 \ldots = \sum_{n=0}^{\infty} (-z)^n$ has radius of convergence $|z| < 1$ and constitutes an expansion around $z = 0$

Expand around $z = i$

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+(z-i)} = \frac{1}{(1+i)\left[1 + \frac{z-i}{1+i}\right]} = \frac{1}{1+i}\left\{1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2}\right.$$

Convergent for $|z-i| < |1+i| = \sqrt{2}$

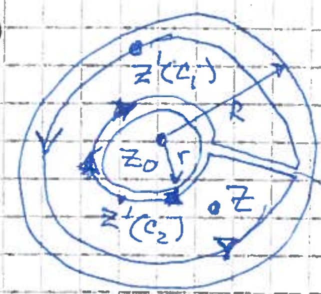
Three representations of the same function with different areas of definition:

1) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$ Maclaurin (around $z=0$)

2) $\frac{1}{1+z} = \frac{1}{1+i}\left\{1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \ldots\right\}$ Taylor around $z = i$

3) $f(z) = \frac{1}{1+z}$ Laurent expansion (one term)

Laurent series



$f(z)$ analytical and single valued in the annular region ("ring") $r < |z| < R$. Define a cut to make it a simply connected region and let C_2 and C_1 be two circles around z_0 with $r < r_2 < r_1 < R$

We have $f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')dz'}{z'-z}$

We want to expand around $z = z_0$ and rewrite the denominator as $z' - z = (z'-z_0) - (z-z_0)$. For C_1 $|z'-z_0| > |z-z_0|$ and for C_2 $|z-z_0| > |z'-z_0|$

Similar to the Taylor expansion we get

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint \frac{f(z')dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}(z'-z_0)^{n-1} f(z')dz'$$

counterclockwise

The first summation is obtained in the same way as the Taylorexpansion while the second is obtained through

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{z'-z} = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)-(z-z_0)} = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{(z-z_0)\left\{\frac{z'-z_0}{z-z_0}-1\right\}} =$$

We have $|z'-z_0| < |z-z_0|$ and expand

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{1}{z-z_0}\sum_{n=0}^{\infty} f(z')\frac{(z'-z_0)^n}{(z-z_0)^n}dz' = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^{-n-1}\oint f(z')(z'-z_0)^n dz'$$

(renumber) $$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2} f(z')(z'-z_0)^{n-1}dz'$$

Collect the contributions: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

where the coefficients $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}$

Note that we could also have done the binomial expansion of $\frac{1}{(z'-z_0)^{n+1}}$ in the ring which eliminates the need for a cut line and the two contours C_1 and C_2. Any contour (counterclockwise) that encircles the inner region will work.

Recommended: example 6.5.1

Singularities: These determine the analytical function in the sense that without any singularity it has to be a constant.

$z = z_0$ is an isolated singular point if $f(z_0)$ is not analytical, but $f(z)$ is analytical in all neighbouring points.

If, in the Laurent expansion $f(z) = \sum_{m=-\infty}^{\infty} a_m (z-z_0)^m$ $a_m = 0$ for $m < -n < 0$ and $a_{-n} \neq 0$, then z_0 is a pole of order n. For $n = 1$ it is a simple pole

If $a_m \neq 0$ for $m \to -\infty$ then the pole is an essential pole where the difference is that for a pole of finite order $(z-z_0)^n f(z)$ is analytic in z_0, while an essential pole can not be regularized.

To examine the behavior as $z \to \infty$ use the change of variables $z \to \frac{1}{t}$ and consider $t \to 0$

Ex: $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \longrightarrow \sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, t^{2n+1}}$

$\Rightarrow t=0$ essential singularity

For $x=0$, i.e. along the imaginary axis

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left(e^{i(iy)} - e^{-i(iy)}\right) = \frac{1}{2i}\left(e^{-y} - e^{y}\right) =$$

$$= i \sinh(y) \longrightarrow \infty \text{ as } y \to \infty$$

$\therefore$ $\sin z$ is not bounded on the complex plane

Branch points and multivalued functions

A complex function gives a mapping of the complex plane onto the complex plane. The mapping is not always 1 to 1.

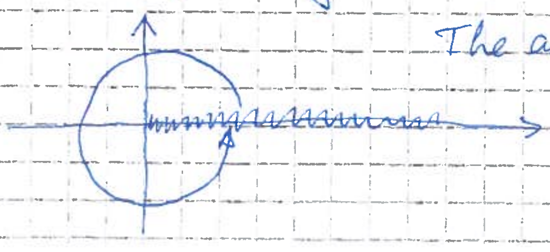
$f(z) = z^a$ where a is not an integer. $(a>0)$

Take z on the unit circle z: $e^{0} \to e^{i2\pi}$

$f(z) \to e^{2\pi i a} \neq e^{0 \cdot a} = 1$, $e^{2\pi i}$ and e^{0} coincide in the z-plane but are mapped onto different points by $f(z)$.

The multivaluedness can be resolved by a cutline joining $z=0$ and $|z| \to \infty$ where the function diverges

The angular interval becomes $[0, 2\pi)$



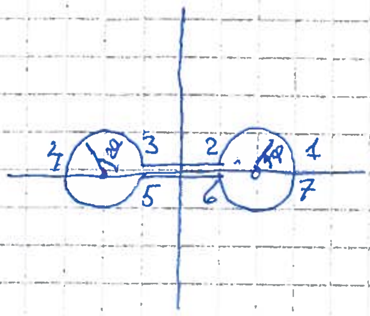
Ex. 6.6.1 $f(z) = (z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$

Branch points at $z = \pm 1$

Simple pole at $z \to \pm\infty$ $\left(z \to \frac{1}{t}\right)$ $(z^2-1)^{1/2} = \left(\frac{1}{t^2}-1\right)^{1/2} =$

$= \frac{1}{t}(1-t^2)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n t^{2n}$

$= \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \ldots$



Take $z+1 = re^{i\vartheta}$

and $z-1 = \varrho e^{i\varphi}$

Total phase is $(\vartheta+\varphi)/2$

Let's follow the phase along $1 \to 7$

Point	ϑ	φ	$(\vartheta+\varphi)/2$
1	0	0	0
2	0	π	$\pi/2$
3	0	π	$\pi/2$
4	π	π	π
5	2π	π	$3\pi/2$
6	2π	π	$3\pi/2$
7	2π	2π	2π

different! (3–5)

different! (2–6)

At 1 & 7 the phase difference is 2π so the function is singlevalued for the chosen contour (encircles both branchpoints). The function remains single-valued ~~for~~ on "arbitrary" contour with cut line $-1 \leq x \leq 1$ or alternatively $x > 1$ and $x < -1$