Green's functions

The Green's function can be viewed as a kernel giving the response to a perturbation

Example: Poisson equation $-\nabla^2 \Psi(\underline{r}) = \frac{1}{\varepsilon_0} \rho(\underline{r})$

The potential at $\underline{r}_1$ from the charge distribution $\rho(\underline{r}_2)$ is

$$\Psi(\underline{r}_1) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\underline{r}_2)}{|\underline{r}_1 - \underline{r}_2|} d^3\underline{r}_2$$ integrating over the entire charge distribution $\rho(\underline{r}_2)$

We identify $G(\underline{r}_1, \underline{r}_2) \equiv \frac{1}{4\pi\varepsilon_0} \frac{1}{|\underline{r}_1 - \underline{r}_2|}$ as the <u>kernel</u> that converts ~~the~~ presence of a charge $\rho(\underline{r}_2) d^3\underline{r}_2$ at $\underline{r}_2$ into a potential at $\underline{r}_1$. We can write

$$\Psi(\underline{r}_1) = \int G(\underline{r}_1, \underline{r}_2) \rho(\underline{r}_2) d^3\underline{r}_2$$ with $G(\underline{r}_1, \underline{r}_2)$ the Green's function.

<u>Dirac δ-function</u> is a distribution with the properties

$\delta(x-t) = 0, \quad t \neq x$ $$\int_a^b \delta(x-t)\,dt = \begin{cases} 1, & a \le x \le b \\ 0, & x \notin [a,b] \end{cases}$$

$$\int_a^b \delta(x-t) f(t)\,dt = \begin{cases} f(x), & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

∴ $f(x)$ can be expanded as $f(x) = \int_a^b \delta(x-t) f(t)\,dt$

Assume a second-order self-adjoint inhomogeneous ODE

$$\mathcal{L} y \equiv \frac{d}{dx}\left\{ p(x) \frac{dy}{dx} \right\} + q(x) y(x) = f(x) \quad \text{for } a \le x \le b$$

and homogeneous boundary conditions, e.g. $y(a) = y(b) = 0$ and/or $y' = 0$ at the end points. $\Rightarrow \mathcal{L}$ Hermitian

Consider a function $G(x,t)$ obtained as the solution of $\mathcal{L} G(x,t) = \delta(x-t)$

(NOTE: in the 6^{th} Ed. $\mathcal{L} G(x,t) = -\delta(x-t)$
Just be consistent!)

With $G(x,t)$ we have for

$$y(x) = \int_a^b G(x,t) f(t)\, dt \quad \text{that} \quad \mathcal{L}_x y(x) = \mathcal{L}_x \int_a^b G(x,t) f(t)\, dt =$$

$$= \int_a^b \mathcal{L}_x G(x,t) f(t)\, dt = \int_a^b \delta(x-t) f(t)\, dt = f(x)$$

so that if we can find $G(x,t)$ we have the solution <u>irrespective</u> of the inhomogeneous right hand side. $G(x,t)$ gives the response at x of the perturbation $f(t)$ at t.

<u>Properties of Green's functions (1-D)</u>

Integrate the (self-adjoint) equation around t with respect to x

$$\int_{t-\varepsilon}^{t+\varepsilon} \frac{d}{dx}\left[p(x)\frac{dG(x,t)}{dx}\right] dx + \int_{t-\varepsilon}^{t+\varepsilon} q(x) G(x,t)\, dx = \underbrace{\int_{t-\varepsilon}^{t+\varepsilon} \delta(t-x)\, dx}_{=1}$$

Both $\frac{dG}{dx}$ and $G(x,t)$ cannot be continuous at $x = t$

($p(x), q(x)$ continuous functions)

Assuming $G(x,t)$ continuous as $x \to t$ $\quad \int_{t-\varepsilon}^{t+\varepsilon} q(x) G(x,t)\, dx \xrightarrow[\varepsilon \to 0]{} 0$

We find a discontinuity in the derivative $\frac{dG(x,t)}{dx}$:

$$\lim_{\varepsilon \to 0+}\left\{\left[\frac{dG(x,t)}{dx}\right]_{x=t+\varepsilon} - \left[\frac{dG(x,t)}{dx}\right]_{x=t-\varepsilon}\right\} = \frac{1}{p(t)}$$

Let $\{\varphi_n(x)\}$ be eigenfunctions of $\mathcal{L}$ (self-adjoint $\rightarrow$ complete orthonormal set)

so that $\mathcal{L}\varphi_n(x) = \lambda \varphi_n(x)$, $\langle \varphi_n | \varphi_m \rangle = \delta_{nm}$

Expand $G(x,t)$ in terms of $\{\varphi_n\}$ as $G(x,t) = \sum_{n,m} g_{nm} \varphi_n(x) \varphi_m^*(t)$

The δ-function $\delta(x-t) = \sum_m \varphi_m(x) \varphi_m^*(t)$

so that $\mathcal{L} \sum_{nm} g_{nm} \varphi_n(x) \varphi_m^*(t) = \sum_m \varphi_m(x) \varphi_m^*(t)$

$\mathcal{L}$ only operates on the x-dependence and $\mathcal{L}\varphi_n(x) = \lambda_n \varphi_n(x)$

$$\Rightarrow \sum_{n,m} \lambda_n g_{nm} \varphi_n(x) \varphi_m^*(t) = \sum_m \varphi_m(x) \varphi_m^*(t)$$

Projecting with $\varphi_n^*(x)\varphi_m(t)$ to find g_{nm} we obtain

$$\lambda_n g_{nm} = \delta_{nm} \quad \text{(n has to be equal to m or the RHS $= 0$)}$$

So that $g_{nm} = \dfrac{\delta_{nm}}{\lambda_n}$ and

$$G(x,t) = \sum_n \frac{\varphi_n(x)\varphi_n^*(t)}{\lambda_n} \quad \text{assuming } \lambda_n \neq 0$$

Thus, $G(x,t) = G(t,x)^*$ symmetry between x and t

<u>Constructing 1-D $G(x,t)$:</u>

Interval $[a,b]$. t is a parameter dividing the interval into two segments $a \le x < t$ and $t < x \le b$

$\mathcal{L} G(x,t) = 0$ for $x < t$ and $x > t$

Seek solutions to the homogeneous equation for $x<t$ satisfying the boundary condition at $x=a$ and for $x>t$ satisfying the boundary condition at $x=b$. Match the solutions at $x=t$ using (i) continuity of $G(x,t)$ and the (controlled) discontinuity of $\dfrac{dG(x,t)}{dx}$ at $x=t$;

Let y_1 solve $\mathcal{L}G(x,t)=0$ and obey the boundary condition at $x=a$ and y_2 similarly for the B.C. at $x=b$.

A general solution for $x<t$ $\quad G_<(x,t) = y_1(x) h_1(t)$

——— " ——— $x>t$ $\quad G_>(x,t) = y_2(x) h_2(t)$

From the symmetry in x and t $\quad G(x,t) = G(t,x)^*$

so $\quad y_1(x) h_1(t) = y_2^*(t) h_2^*(x) \Rightarrow h_2^*(x) = y_1(x) \dfrac{h_1(t)}{y_2^*(t)}$

$$\Rightarrow \frac{h_1(t)}{y_2^*(t)} = A \text{ (constant) and } h_2^*(x) = y_1(x) \cdot A$$

so $G(x,t) = \begin{cases} A\, y_1(x)\, y_2(t) & x<t \\ A\, y_2(x)\, y_1(t) & x>t \end{cases}$ Assume y_1, y_2 real

The discontinuity in the derivative at $x=t$

$$\left.\frac{dG_>}{dx}\right|_{x=t} - \left.\frac{dG_<}{dx}\right|_{x=t} = \frac{1}{p(t)} \quad \text{becomes} \quad A\overbrace{\left\{ y_2'(t)\,y_1(t) - y_1'(t)\,y_2(t) \right\}}^{\text{Wronskian}} = \frac{1}{p(t)}$$

so that $A = \left\{ p(t) \cdot W\{y_1(t), y_2(t)\} \right\}^{-1}$

Explicit expresson for $y(x)$:

$$y(x) = \int_a^x G_>(x,t) f(t)\,dt + \int_x^b G_<(x,t) f(t)\,dt =$$

$$= A\, y_2(x) \int_a^x y_1(t) f(t)\,dt + A\, y_1(x) \int_x^b y_2(t) f(t)\,dt$$

For $x \to a$ the first integral becomes zero and the second term is proportional to $y_1(a)$. Similar for $x \to b$.

To show that $\mathcal{L}\,y(x) = f(x)$ for this $y(x)$ note that

$$\frac{d}{dx}\left\{ y_2(x) \int_a^x y_1(t) f(t)\,dt \right\} = y_2' \int_a^x y_1(t) f(t)\,dt + y_2(x)\, y_1(x)\, f(x)$$

and $\frac{d}{dx}\left\{ y_1(x) \int_x^b y_2(t) f(t)\,dt \right\} = y_1' \int_x^b y_2(t) f(t)\,dt - y_1(x)\, y_2(x)\, f(x)$

Example 10.1.1. $-y'' = f(x)$ with $y(0) = y(1) = 0$ $x \in [0,1]$

Solutions to $y'' = 0$ are $y(x) = c_0 + c_1 x$

Take $y_1 = x$ so that $y_1(0) = 0$

$y_2 = 1 - x$ — " — $y_2(1) = 0$

The ODE is on $\underbrace{\text{standard form}}_{\text{(self-adjoint)}}$ with $p(x) = -1$

so that $A = \dfrac{1}{(-1)\left[(-1)x - (1)(1-x)\right]} = 1$

so that $G(x,t) = \begin{cases} x(1-t), & 0 \le x < t \\ (1-x)\,t; & t < x \le 1 \end{cases}$

Set $f(x) = 2x$; $y(x) = (1-x)\int_0^x t \cdot 2t\,dt + x\int_x^1 (1-t)\,2t\,dt =$

$$= (1-x)\frac{2}{3}x^3 + x\left[t^2 - \frac{2}{3}t^3\right]_x^1 = \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{1}{3}x - x^3 + \frac{2}{3}x^4 =$$

$$= \frac{1}{3}x - \frac{1}{3}x^3 \qquad y(0) = 0,\ y(1) = 0$$

$$y' = \frac{1}{3} - x^2, \quad -y'' = 2x \qquad \text{OK}$$

Set $f(x) = \sin \pi x$; $y(x) = (1-x)\int_0^x t \sin \pi t\,dt + x\int_x^1 (1-t)\sin \pi t\,dt$

$$\int_0^x t\sin\pi t\,dt = \left[-\frac{t}{\pi}\cos\pi t\right]_0^x + \frac{1}{\pi}\int_0^x \cos\pi t\,dt = -\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\left[\sin\pi t\right]_0^x =$$

$$= -\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x$$

$$\int_x^1 (1-t)\sin\pi t\,dt = \int_x^1 \sin\pi t\,dt - \int_x^1 t\sin\pi t\,dt = -\frac{1}{\pi}\left[\cos\pi t\right]_x^1 -$$

$$-\left\{\left[-\frac{t}{\pi}\cos\pi t\right]_x^1 + \frac{1}{\pi}\int_x^1 \cos\pi t\,dt\right\} =$$

$$= -\cancel{\frac{1}{\pi}\cos\pi} + \frac{1}{\pi}\cos\pi x + \cancel{\frac{1}{\pi}\cos\pi} - \frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x$$

so that $y(x) = (1-x)\left\{-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right\} + x\left\{\frac{1}{\pi}\cos\pi x - \frac{x}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x\right\} =$

$$= -\frac{x}{\pi}\cos\pi x + \frac{x^2}{\pi}\cos\pi x + \frac{1}{\pi^2}\sin\pi x - \frac{x}{\pi^2}\sin\pi x +$$

$$+ \frac{x}{\pi}\cos\pi x - \frac{x^2}{\pi}\cos\pi x + \frac{x}{\pi^2}\sin\pi x = \frac{1}{\pi^2}\sin\pi x$$

$$y(0) = y(1) = 0 \qquad \text{and } -y'' = \sin\pi x$$

∴ The Green's function can be used to construct the solution for arbitrary right hand side (assuming we can do the integrals)

<u>Non-homogeneous boundary conditions</u> $y(a) = c_1,\ y(b) = c_2$

c_1 and/or $c_2 \neq 0$

set $$u = y - \frac{c_1(b-x) + c_2(x-a)}{b-a}$$

$$\Rightarrow u(a) = y(a) - c_1 = 0$$

$$u(b) = y(b) - c_2 = 0$$

Initial value problems: Take $\mathcal{L}y = y'' + y = f(x)$
with $y(0) = y'(0) = 0$; $p(x) = 1$

Linearly independent solutions $y_1 = \sin x$, $y_2 = \cos x$

but $c_1 \sin x + c_2 \cos x$ gives $c_2 = 0$ at $x = 0$

Derivative $c_1 \cos x - c_2 \sin x$ gives $c_1 = 0$ at $x = 0$

So for $x < t$ $G(x,t) = 0$

For $x > t$ no constraining B.C. so set

$$G(x,t) = c_1(t) \sin x + c_2(t) \cos x \,, \quad x > t$$

Continuity of $G(x,t)$ ~~the solution~~ as $x \to t_{+,-}$ $0 = c_1(t)\sin t + c_2(t)\cos t$

Discontinuity of the derivative

$$\frac{\partial G}{\partial x}(t_+, t) - \frac{\partial G(t_-, t)}{\partial x} = \frac{1}{p(t)} = 1 \rightarrow c_1(t)\cos t - c_2(t)\sin t = 1$$

We have $c_1(t) = -c_2(t)\dfrac{\cos t}{\sin t} \rightarrow -c_2(t)\left\{\dfrac{\cos^2 t}{\sin t} + \sin t\right\} = 1$

and $c_2(t) = -\sin t$, $c_1(t) = \cos t$

Thus, $G(x,t) = \sin x \cos t - \cos x \sin t = \sin(x-t)$, $x > t$

and
$$G(x,t) = \begin{cases} 0 \,, & x < t \\ \sin(x-t) \,, & t < x \end{cases}$$

$$y(x) = \int_0^\infty G(x,t) f(t)\,dt = \int_0^x \sin(x-t) f(t)\,dt$$

$G(x,t)$ not symmetric since B.C. only on one side

Boundary at infinity

$\left(\dfrac{d^2}{dx^2} + k^2\right)\psi(x) = g(x)$ with solutions as outgoing waves at infinity $y_1 = e^{-ikx}$, $y_2 = e^{ikx}$

$\psi(x,t) = e^{-i\omega t} \cdot y_i \rightarrow \underset{y_1}{e^{-i(kx+\omega t)}}, \underset{y_2}{e^{i(kx-\omega t)}}$

Outgoing wave $\rightarrow$ y_2 for large, positive x

y_1 for large, negative x

So

$$G(x,x') = \begin{cases} A\, y_1(x')\, y_2(x) \,, & x > x' \\ A\, y_2(x')\, y_1(x) \,, & x < x' \end{cases}$$

$$p(x) = 1 \implies A = \frac{1}{y_2' y_1 - y_1' y_2} = \frac{1}{ik + ik} = -\frac{i}{2k}$$

and $G(x,x') = -\frac{i}{2k} \exp\left(i\,|x-x'|\right)$

Integral equations

Consider the eigenvalue equation $\mathcal{L} y(x) = \lambda y(x)$ with $\mathcal{L}$ self-adjoint and $y(a) = y(b) = 0$ as an inhomogeneous ODE with $\lambda y(x)$ as the right hand side. Find the Green's function $G(x,t)$ and write $y(x) = \lambda \int_a^b G(x,t)\, y(t)\, dt$

The ODE is converted to an integral equation.

It ~~solves~~ satisfies the ODE since

$$\mathcal{L}_x y(x) = \lambda \mathcal{L}_x \int_a^b G(x,t)\, y(t)\, dt = \lambda \int_a^b \mathcal{L}_x G(x,t)\, y(t)\, dt =$$

$$= \lambda \int_a^b \delta(x-t)\, y(t)\, dt = \lambda y(x)$$

$G(x,t)$ kernel with boundary conditions built in

Problems in two and three dimensions

- A homogeneous PDE $\mathcal{L}\psi(\underline{r}_1) = 0$ and its boundary conditions define a Green's function $G(\underline{r}_1, \underline{r}_2)$ as the solution to $\mathcal{L}G(\underline{r}_1, \underline{r}_2) = \delta(\underline{r}_1 - \underline{r}_2)$ subject to the relevant B.C.

- The solution $\psi(\underline{r})$ to $\mathcal{L}\psi(\underline{r}) = f(\underline{r})$ can be written
$$\psi(\underline{r}_1) = \int G(\underline{r}_1, \underline{r}_2) f(\underline{r}_2)\, d^3\underline{r}_2$$
integrating over the entire space relevant to the problem

- When $\mathcal{L}$ with its B.C. define $\mathcal{L}\psi = \lambda\psi$ as Hermitian eigenvalue problem with eigenfctns $\varphi_n(x)$ and eigenvalues λ_n then
$$G(\underline{r}_1, \underline{r}_2) = G^*(\underline{r}_2, \underline{r}_1) \quad \text{symmetric}$$
and
$$G(\underline{r}_1, \underline{r}_2) = \sum_n \frac{\varphi_n^*(\underline{r}_2)\varphi_n(\underline{r}_1)}{\lambda_n} \quad \text{eigenfunction expansion}$$

- $G(\underline{r}_1, \underline{r}_2)$ continuous and differentiable for $\underline{r}_1 \neq \underline{r}_2$

Self-adjointness

$$\mathcal{L}\psi(\underline{r}) = \nabla\left[p(\underline{r})\nabla\psi(\underline{r})\right] + q(\underline{r})\psi(\underline{r}) = f(\underline{r})$$

Eigenfunction expansion with parameter λ

Write $\mathcal{L}\psi(\underline{r}_1) = \lambda\psi(\underline{r})$ as $\mathcal{L}\psi(\underline{r}) - \lambda\psi(\underline{r}) = 0$

For the Green's function $\mathcal{L}\psi(\underline{r}_1) - \lambda\psi(\underline{r}_1) = \delta(\underline{r}_2 - \underline{r}_1)$

Eigenfunction expansion becomes $G(\underline{r}_1, \underline{r}_2) = \sum_n \frac{\varphi_n^*(\underline{r}_2)\varphi_n(\underline{r}_1)}{\lambda_n - \lambda}$

The eigenvalues are found as poles of $G(\underline{r}_1, \underline{r}_2)$ (divergence).

Specific forms

Laplace operator in 3-D with B.C. that G vanish at infinity

$$\nabla_1^2 G(\underline{r}_1, \underline{r}_2) = \delta(\underline{r}_1 - \underline{r}_2) \quad , \quad \lim_{|\underline{r}_1| \to \infty} G(\underline{r}_1, \underline{r}_2) = 0$$

B.C. spherically symmetric and at infinity so we can assume $G(\underline{r}_1, \underline{r}_2)$ to be a function of $r_{12} = |\underline{r}_1 - \underline{r}_2|$

Integrate over spherical volume of radius a centered at $\underline{r}_2$

$$\int_{r_{12}<a} \nabla_1 \cdot \nabla_1 G(\underline{r}_1, \underline{r}_2)\, d^3 r_1 = 1 \leftarrow \text{from } \delta\text{-function}$$

Apply Gauss' theorem $\displaystyle\int_{r_{12}=a} \nabla_1 G(\underline{r}_1, \underline{r}_2)\, d\underline{\sigma}_1 = 4\pi a^2 \frac{dG}{dr_{12}}\bigg|_{r_{12}=a} = 1$

$$\therefore \quad \frac{dG(\underline{r}_1, \underline{r}_2)}{dr_{12}} = \frac{1}{4\pi r_{12}^2} \quad \to G(\underline{r}_1, \underline{r}_2) = -\frac{1}{4\pi}\frac{1}{|\underline{r}_1 - \underline{r}_2|}$$

Satisfies the B.C. at ∞
(otherwise a constant of integration can be added)

For different B.C. a suitable solution to the homogeneous equation can be added.

2-D: circular coordinates $\underline{\rho} = (\rho, \varphi)$ and infinite extent

$$\int_{\rho_{12}=a} \nabla_1 G(\underline{\rho}_1, \underline{\rho}_2)\, d\underline{\sigma}_1 = 2\pi a \frac{dG}{d\rho_{12}}\bigg|_{\rho_{12}=a} = 1 \quad \to \frac{dG}{d\rho_{12}} = \frac{1}{2\pi\rho_{12}}$$

$$\text{and} \quad G(\underline{\rho}_1, \underline{\rho}_2) = \frac{1}{2\pi} \ln|\underline{\rho}_1 - \underline{\rho}_2|$$

Table 10.1 3-D

	Laplace	Helmholtz	Modified Helmholtz
Operator	∇^2	$\nabla^2 + k^2$	$\nabla^2 - k^2$
$G(\underline{r}_1, \underline{r}_2)$	$-\frac{1}{4\pi}\frac{1}{\lvert\underline{r}_1 - \underline{r}_2\rvert}$	$-\frac{\exp(ik\lvert\underline{r}_1 - \underline{r}_2\rvert)}{4\pi\lvert\underline{r}_1 - \underline{r}_2\rvert}$	$-\frac{\exp(-k\lvert\underline{r}_1 - \underline{r}_2\rvert)}{4\pi\lvert\underline{r}_1 - \underline{r}_2\rvert}$
B.C.	$G \xrightarrow[r_{12}\to\infty]{} 0$	outgoing wave	$G \xrightarrow[r_{12}\to\infty]{} 0$

Accommodating B.C.

1-D Laplace eq. $\frac{d^2\psi(x)}{dx^2} = 0$ with $G(x_1,x_2) = \frac{1}{2}|x_1-x_2|$

we want $\psi(0) = \psi(1) = 0$. We can add terms of the form $f(x_1)g(x_2)$ ~~with~~ where $\frac{d^2 f}{dx_1^2} = 0$ and $\frac{d^2 g}{dx_2^2} = 0$ without affecting the continuity of G or the discontinuity of G'. f and g will be of the form $ax+b$

Take $G(x_1,x_2) = -\frac{1}{2}(x_1+x_2) + x_1x_2 + \frac{1}{2}|x_1-x_2| =$

$$= \begin{cases} -\frac{1}{2}(x_1+x_2) + x_1x_2 + \frac{1}{2}(x_2-x_1), & x_1 < x_2 \\ -\frac{1}{2}(x_1+x_2) + x_1x_2 + \frac{1}{2}(x_1-x_2), & x_2 < x_1 \end{cases}$$

$$G(0,x_2) = -\frac{1}{2}x_2 + \frac{1}{2}x_2 = 0$$

$$G(1,x_2) = -\frac{1}{2}(1+x_2) + x_2 + \frac{1}{2}(1-x_2) = 0$$

Quantum Mechanical Scattering - Born Approximation

A beam of particles is moving along the negative z-axis towards a scattering potential $V(\underline{r})$ at the origin where a small fraction is scattered and goes out as a spherical wave

The Schrödinger eq. $-\frac{\hbar^2}{2m}\nabla^2\psi(\underline{r}) + V(\underline{r})\psi(\underline{r}) = E\psi(\underline{r})$ can be rearranged as

$$\nabla^2\psi(\underline{r}) + k^2\psi(\underline{r}) = \left[\frac{2m}{\hbar^2}V(\underline{r})\psi(\underline{r})\right] \quad \text{with } k^2 = \frac{2mE}{\hbar^2}$$

Look for solutions having the asymptotic form

$$\psi(\underline{r}) \sim e^{i\underline{k}_0\cdot\underline{r}} + f_k(\vartheta,\varphi)\frac{e^{ik\cdot r}}{r}$$

spherical wave

incident plane wave

scattering amplitude gives cross-section $|f_k(\vartheta,\varphi)|^2$

elastic scattering $|\underline{k}_0| = k$

Write the solution as

$$\psi(\underline{r}_1) = \int \frac{2m}{\hbar^2}V(\underline{r}_2)\psi(\underline{r}_2)G(\underline{r}_1,\underline{r}_2)\,d^3\underline{r}_2$$

Use the Green's function for the Helmholtz operator

$$G(\underline{r}_1, \underline{r}_2) = -\frac{1}{4\pi} \frac{e^{ik|\underline{r}_1-\underline{r}_2|}}{|\underline{r}_1-\underline{r}_2|}$$

which gives the correct asymptotic form

Add incident wave $e^{i\underline{k}_0\cdot\underline{r}}$ (solution to the homogeneous eq.)

Thus, $$\psi(\underline{r}_1) = e^{i\underline{k}_0\cdot\underline{r}_1} - \int \frac{2m}{\hbar^2} V(\underline{r}_2)\psi(\underline{r}_2) \frac{e^{ik|\underline{r}_1-\underline{r}_2|}}{4\pi|\underline{r}_1-\underline{r}_2|} d^3\underline{r}_2 \quad \text{(exact)}$$

Lippmann - Schwinger eq.

For weak scattering, such that $e^{i\underline{k}_0\cdot\underline{r}_1}$ dominates the solution we can approximate $\psi(\underline{r}_2)$ in the integral by $e^{i\underline{k}_0\cdot\underline{r}_2}$ and find the Born approximation (1st order)

$$\psi_1(\underline{r}_1) = e^{i\underline{k}_0\cdot\underline{r}_1} - \int \frac{2m}{\hbar^2} V(\underline{r}_2) \frac{e^{ik|\underline{r}_1-\underline{r}_2|}}{4\pi|\underline{r}_1-\underline{r}_2|} e^{i\underline{k}_0\cdot\underline{r}_2} d^3\underline{r}_2$$

The second order is obtained by using $\psi_1(\underline{r}_1)$ in the Lippmann-Schwinger equation

$$\psi_2(\underline{r}_1) = e^{i\underline{k}_0\cdot\underline{r}_1} - \int \frac{2m}{\hbar^2} V(\underline{r}_2) \frac{e^{ik|\underline{r}_1-\underline{r}_2|}}{4\pi|\underline{r}_1-\underline{r}_2|} \psi_1(\underline{r}_2) d^3\underline{r}_2 =$$

$$= \psi_1(\underline{r}_1) + \left(\frac{2m}{\hbar^2}\right)^2 \int V(\underline{r}_2) \frac{e^{ik|\underline{r}_1-\underline{r}_2|}}{4\pi|\underline{r}_1-\underline{r}_2|} d^3\underline{r}_2 \int V(\underline{r}_3) \frac{e^{ik|\underline{r}_2-\underline{r}_3|}}{4\pi|\underline{r}_2-\underline{r}_3|} e^{i\underline{k}_0\cdot\underline{r}_3} d^3\underline{r}_3$$

and so on