Ordinary Differential Equations (ODE)

Differentiating is a linear operation

$$\frac{d}{dx}\left(a\varphi(x)+b\psi(x)\right)=a\frac{d\varphi}{dx}+b\frac{d\psi}{dx}=a\mathcal{L}\varphi+b\mathcal{L}\psi=$$

$$=\mathcal{L}(a\varphi+b\psi) \text{ with } \mathcal{L}\equiv\frac{d}{dx} \quad (\text{operator})$$

Linearity is a property of the operator, e.g. $\mathcal{L}=p(x)\frac{d}{dx}+q(x)$

$$\mathcal{L}(a\varphi(x)+b\psi(x))=a\left(p(x)\frac{d\varphi}{dx}+q(x)\varphi(x)\right)+b\left(p(x)\frac{d\psi}{dx}+q(x)\psi(x)\right)$$

$$=a\mathcal{L}\varphi(x)+b\mathcal{L}\psi(x)$$

No requirements on $p(x)$ or $q(x)$ except being multiplicative functions of x.

General linear operator $\mathcal{L}\equiv\sum_{\nu=0}^{n}P_\nu(x)\left(\frac{d^\nu}{dx^\nu}\right)$

Definitions: Homogeneous ODE – dependent variable $(\varphi(x))$ occurs to the same power in all its terms $\mathcal{L}\varphi(x)=0$

Inhomogeneous otherwise, e.g. $\mathcal{L}\varphi(x)=F(x)$

Linear if it can be written as $\mathcal{L}\varphi(x)=F(x)$ with $F(x)$ an algebraic function of x

Linearity gives superposition principle, i.e. $\mathcal{L}\varphi=0$, $\mathcal{L}\psi=0$
$\Rightarrow \mathcal{L}(a\varphi+b\psi)=0$

Bernoulli equation $y'=p(x)y+q(x)y^n$, $n\neq 0,1$ is an example of a nonlinear differential equation (cannot be written as $\mathcal{L}y$)

First-order equations (highest derivative is $y'=\frac{dy}{dx}$)

General form $\frac{dy}{dx}=f(x,y)=-\frac{P(x,y)}{Q(x,y)}$

Separable equations: $\frac{dy}{dx}=-\frac{P(x)}{Q(y)} \rightarrow P(x)dx+Q(y)dy=0$

$$\Rightarrow \int_{x_0}^{x}P(x)dx+\int_{y_0}^{y}Q(y)dy=0$$

The equation $P(x,y)dx + Q(x,y)dy = 0$ is exact if the LHS can be matched to a differential $d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$ with

$$\frac{\partial\varphi}{\partial x} = P(x,y) \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = Q(x,y)$$

$\varphi(x,y)$ exists requires $\frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} = \frac{\partial^2\varphi}{\partial x\partial y}$

We find the solution by integration

$$\int_{x_0}^{x} \frac{\partial\varphi}{\partial x}dx = \int_{x_0}^{x} P(x,y)dx \longrightarrow \varphi(x,y) - \varphi(x_0,y) = \int_{x_0}^{x} P(x,y)dx$$

$$\frac{\partial\varphi}{\partial y} = Q(x,y) = \int_{x_0}^{x} \frac{\partial P(x,y)}{\partial y}dx + \frac{\partial\varphi(x_0,y)}{\partial y} =$$

$$= \int_{x_0}^{x} \frac{\partial Q(x,y)}{\partial x}dx + \frac{\partial\varphi(x_0,y)}{\partial y} = Q(x,y) - Q(x_0,y) + \frac{\partial\varphi(x_0,y)}{\partial y}$$

$$\therefore \frac{\partial\varphi(x_0,y)}{\partial y} = Q(x_0,y) \longrightarrow \varphi(x_0,y) = \int_{y_0}^{y} Q(x_0,y)dy + \varphi(x_0,y_0)$$

$$\Rightarrow \varphi(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = \text{constant}$$

Exact equations not necessarily separable:

$$y' + \left(1 + \frac{y}{x}\right) = 0 \longrightarrow \underbrace{(x+y)}_{P(x,y)}dx + \underbrace{x}_{Q(x,y)}dy = 0$$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(x+y) = 1 \\ \frac{\partial Q}{\partial x} &= \frac{d}{dx}(x) = 1 \end{aligned}\right\} \text{exact}$$

$$\varphi(x,y) = \int_{x_0}^{x} (x+y)dx + \int_{y_0}^{y} x_0\,dy = \left\{\tfrac{1}{2}x^2 + xy - \tfrac{1}{2}x_0^2 - x_0 y\right\} + \left\{x_0 y - x_0 y_0\right\} =$$

$$= \tfrac{1}{2}x^2 + xy + \text{constant terms}$$

$$\therefore \varphi(x,y) = \tfrac{1}{2}x^2 + xy = C \Rightarrow y = \frac{C}{x} - \frac{1}{2}x$$

Check: $y' = -\frac{C}{x^2} - \frac{1}{2}$

$$y' + 1 + \frac{y}{x} = -\frac{C}{x^2} - \frac{1}{2} + 1 + \frac{C}{x^2} - \frac{1}{2} = 0$$

Equations are said to be homogeneous of order n in x and y if the combined powers of x and y add to n in all the terms of $P(x,y)$ and $Q(x,y)$ in $P(x,y)dx + Q(x,y)dy = 0$

$(2x+y)dx + x\,dy = 0$ homogeneous in x and y

substitute $y = xv$, $dy = x\,dv + v\,dx$

$(2v+2)dx + x\,dv = 0 \rightarrow \frac{dx}{x} + \frac{1}{2}\frac{dv}{v+1} = 0$

$\ln x + \frac{1}{2}\ln(v+1) = C \rightarrow x^2(v+1) = C \rightarrow x(\underbrace{xv}_{y} + x) = C$

$y = \frac{C}{x} - x$

Isobaric equations: Assigning x and dx weight 1 and y and dy weight m makes the equation homogeneous (of some order) then $y = x^m v$ makes the equation separable

$(x^2 - y)dx + x\,dy = 0$ $\quad x^2dx$ weight 3, $\quad x\,dy,\ y\,dx$ weight $m+1$ $\Big\}\ m = 2$

$y = x^2 v \qquad dy = 2xv\,dx + x^2 dv$

$(x^2 - x^2v)dx + x(2xv\,dx + x^2dv) = x^2\{(1-v+2v)dx + x\,dv\} = 0$

$(v+1)dx + x\,dv = 0 \rightarrow \ln x + \ln(v+1) = \ln C \rightarrow x(v+1) = C$

$v = \frac{C}{x} - 1 \qquad y = x^2 v \rightarrow y = Cx - x^2$

Integrating factor:

General procedure exists for $\frac{dy}{dx} + p(x)y = q(x)$

Seek integrating factor $\alpha(x)$ so that

$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x)$ can be written

$\frac{d}{dx}\left[\alpha(x)y\right] = \alpha(x)q(x)$;

This requires $\alpha(x)\frac{dy}{dx} + \frac{d\alpha}{dx}y = \alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y$

$\Rightarrow \frac{d\alpha}{dx} = \alpha(x)p(x)$

We separate and integrate

$$\int^{\alpha} \frac{d\alpha}{\alpha} = \int^{x} p(x)\,dx \rightarrow \alpha(x) = \exp\left[\int^{x} p(x)\,dx\right]$$

The constant can be left out since adding a constant only gives a multiple of the equation: $(\alpha(x)+c)\,dy = (\alpha(x)+c)\,q(x$

Integrate $\frac{d}{dx}\left[\alpha(x)y\right] = \alpha(x)q(x)$ to obtain $y(x) = \frac{1}{\alpha(x)}\left[\int^{x}\alpha(x)q(x)\,dx + C\right] \equiv y_2(x) + y_1(x)$

$y_1(x) = \frac{C}{\alpha(x)}$ solves homogeneous eq. $(q(x)=0)$

$y_2(x) = \frac{1}{\alpha(x)}\int^{x}\alpha(x)q(x)\,dx$ contains the source $q(x)$ and is the particular solution.

- The solution of an inhomogeneous first-order linear ODE is unique except for an arbitrary multiple of the solution to the corresponding homogeneous ODE
- A first-order linear homogeneous ODE has only one linearly independent solution.

<u>Variation of the constant</u>

Solve first the homogeneous equation $y' + p(x)y = 0$

$$\frac{y'}{y} = -p(x) \rightarrow \ln y = -\int^{x} p(X)\,dX + \ln C \rightarrow y(x) = C\exp\left[-\int^{x} p(X)\,dX\right]$$

Allow C to be a function of x $\quad C \rightarrow C(x)$

$$y' = \exp\left[-\int^{x} p(X)\,dX\right]\left\{C'(x) + C(x)\frac{d}{dx}\left\{-\int^{x} p(X)\,dX\right\}\right\} =$$

$$= \exp\left[-\int^{x} p(X)\,dX\right]\left\{C'(x) - p(x)C(x)\right\}$$

Substitute into $y' + py = q$

$$C'(x)\exp\left[-\int^{x} p(X)\,dX\right] = q$$

$$\Rightarrow C(x) = \int^{x}\exp\left[\int^{X} p(Y)\,dY\right] q(X)\,dX \Rightarrow y = C(x)\exp\left[-\int^{x} p(X)\,dX\right]$$

ODE's with constant coefficients

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = F(x)$$ Any order n
a_i constants

Homogeneous equation has solutions e^{mx} where m solves $m^n + a_{n-1}m^{n-1} + \dots + a_1 m + a_0 = 0$

In the case of a double root consider e^{mx}, $\lim_{\Delta\to 0} e^{(m+\Delta)x}$.
Since both solve the homogeneous equation so does
$$\lim_{\Delta\to 0} \frac{e^{(m+\Delta)x} - e^{mx}}{\Delta} = x e^{mx}$$. For a triple root $e^{mx}, xe^{mx}, x^2 e^{mx}$

Second-order linear ODE's

Singular points $\qquad y'' + P(x)y' + Q(x)y = 0$

A point x_0 is an <u>ordinary</u> point if both $P(x_0)$ and $Q(x_0)$ are finite

If either $P(x)$ or $Q(x)$ diverge as $x \to x_0$ then x_0 is a <u>singular</u> point.

The singular point x_0 is <u>regular</u> if either $P(x)$ or $Q(x)$ diverge there but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remain finite

The singular point x_0 is <u>irregular</u> if $P(x)$ diverges faster than $(x-x_0)^{-1}$ so that $(x-x_0)P(x) \to \infty$ as $x \to x_0$ or if $Q(x)$ diverges faster than $(x-x_0)^{-2}$ so that $(x-x_0)^2 Q(x) \to \infty$ as $x \to x_0$

<u>Point at ∞</u>: Set $x = 1/z$ and investigate for $z \to 0$

New dependent variable $w(z) = y(z^{-1})$

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = \frac{dw(z)}{dz}\left(-\frac{1}{x^2}\right) = -z^2 w'$$

$$y'' = \frac{dy'}{dz}\frac{dz}{dx} = (-z^2)\frac{d}{dz}\left(-z^2 w'\right) = z^4 w'' + 2z^3 w'$$

Substitute in ODE: $z^4 w'' + 2z^3 w' - z^2 P(z^{-1}) w' + Q(z^{-1}) w = 0$

Put equation on normal form:

$$w'' + \left[\frac{2z^3 - z^2 P(z^{-1})}{z^4}\right] w' + \frac{Q(z^{-1})}{z^4} w = 0$$

Behavior determined by $\frac{2z - P(z^{-1})}{z^2}$ and $\frac{Q(z^{-1})}{z^4}$. If they are finite at $z=0$ then $z=0$ $(x \to \infty)$ is a regular point. If they diverge no more rapidly than $1/z$ and $1/z^2$ respectively then $x=\infty$ is a regular singular point, otherwise irregular (essential singularity).

Hermite eq. $y'' - 2xy' + 2\alpha y = 0$ No finite singularities

$x \to \infty$; $x \to 1/z$ $\quad P(z^{-1}) = -\frac{2}{z}$, $Q(z^{-1}) = 2\alpha$

$$\frac{2z + 2/z}{z^2} = 2\,\frac{z^2+1}{z^3}$$

$z=0$ $(x=\infty)$ irregular singularity

Series Solutions – Frobenius' Method

Make the Ansatz $\quad y(x) = \sum_{j=0}^{\infty} a_j x^{j+s} \equiv x^s \sum_{j=0}^{\infty} a_j x^j$

We require $a_0 \neq 0$ so that x^s is the leading term as $x \to$ (expansion around $x=0$). Any point (not essential singularity) can be chosen, e.g. $y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{j+s}$ expansion around x_0

Take derivatives

$$y' = \sum_{j=0}^{\infty} (j+s)\, a_j\, x^{j+s-1}$$

$$y'' = \sum_{j=0}^{\infty} (j+s)(j+s-1)\, a_j\, x^{j+s-2}$$

and introduce in the ODE. Arrange terms to have equal powers of x and use the fact that $\{x^n\}$ are linearly independent, i.e. $\sum_{k=0}^{n} a_k x^k = 0 \Rightarrow a_k = 0$

Example: Linear oscillator $\quad y'' + \omega^2 y = 0$

$$\sum_{j=0}^{\infty} (j+s)(j+s-1)\, a_j\, x^{j+s-2} + \omega^2 \sum_{k=0}^{\infty} a_k x^{j+s} = 0$$

The lowest powers x^{s-2} and x^{s-1} only occur in the first sum. Rewrite

$$s(s-1)a_0 x^{s-2} + (s+1)s\, a_1 x^{s-1} + \sum_{j=0}^{\infty} x^{j+s}\{a_{j+2}(j+2+s)(j+s+1) + \omega^2 a_j\} = 0$$

With $a_0 \neq 0$ we have the indicial equation $s(s-1)=0$ $\begin{cases} s=0 \\ s=1 \end{cases}$

For $s=1$ a_1 must be zero, but for $s=0$ a_1 can be non-zer

The sum gives a recurrence relation:

$$a_{j+2}(j+2+s)(j+s+1) + \omega^2 a_j = 0$$

or

$$a_{j+2} = -\frac{\omega^2}{(j+2+s)(j+s+1)} a_j$$

convergent since $\lim_{j\to\infty} \left|\frac{a_{j+2}}{a_j}\right| \sim \frac{1}{j^2}$

$s=1$:

$$a_{j+2} = -\frac{\omega^2}{(j+3)(j+2)} a_j$$

Take the first few terms back to a_0 to find a pattern

$j=0$ $\quad a_2 = -\frac{\omega^2}{3\cdot 2} a_0$

$j=2$ $\quad a_4 = -\frac{\omega^2}{5\cdot 4} a_2 = \frac{\omega^4}{5\cdot 4\cdot 3\cdot 2} a_0$

$j=3$ $\quad a_6 = -\frac{\omega^2}{7\cdot 6} a_4 = -\frac{\omega^6}{7!} a_0$

General: $a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!}$ and $y(x)_{s=1} = a_0 x \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n+1)!} =$

$$= \frac{a_0}{\omega} \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!} = \frac{a_0}{\omega} \sin \omega x$$

$s=0$: a_1 is undetermined, but there is no connection from a_0 to odd terms. Set $a_1 = 0$ which gives $a_3 = a_5 = \dots 0$

We have $a_{j+2} = -\frac{\omega^2}{(j+2)(j+1)} a_j$

$j=0$ $\quad a_2 = -\frac{\omega^2}{2\cdot 1} a_0$

$j=2$ $\quad a_4 = -\frac{\omega^2}{4\cdot 3} a_2 = \frac{\omega^4}{4!} a_0$

General: $a_{2n} = (-1)^n \frac{\omega^{2n}}{2n!}$ $\Rightarrow$ $y(x)_{s=0} = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n)!} = a_0 \cos \omega x$

Scheme: $\boxed{a_0 s(s-1)}_{=0} x^{s-2} + \boxed{a_1 s(s+1)}_{=0} x^{s-1} + \boxed{\begin{matrix} a_2(s+2)(s+1) \\ a_0 \omega^2 \end{matrix}}_{=0} x^{s} + \boxed{\begin{matrix} a_3(s+3)(s+2) \\ a_1 \omega^2 \end{matrix}}_{=0} x^{s+1}$ 8

The acceptability of a series solution depends on its convergence

We obtained two solutions, one of even $y_1(x) = y_1(-x)$ and one of odd $y_2(x) = -y_2(-x)$ ($\cos \omega x$ and $\sin \omega x$)

$\mathcal{L}(x) = \frac{d^2}{dx^2} + \omega^2$ is even under the parity operation

$\mathcal{L}(-x) = \mathcal{L}(x)$. When $\mathcal{L}$ has definite parity $+x$ and $-x$ may be interchanged and $\mp \mathcal{L}(x) y(-x) = 0$ then if $y(x)$ is a solution, also $y(-x)$ satisfies the ODE. We can always then obtain solutions as even or odd functions through

$$y_{even}(x) = y(x) + y(-x) \quad , \quad y_{odd} = y(x) - y(-x)$$

Legendre, Chebyshev, Bessel, simple harmonic oscillator and Hermite equations have $P(x)$ odd and $Q(x)$ even and thus preserve parity $[\, y'' + P(x) y' + Q(x) y = 0 \,]$

Laguerre differential operator $y'' + \frac{1-x}{x} y' + a y = 0$ has indefinite parity.

Bessel equation — only one solution found

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

Ansatz: $y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

$$\Rightarrow \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} -$$

$$- n^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

Lowest power x^k $(\lambda = 0)$ $\Rightarrow$ $a_0 \{ k(k-1) + k - n^2 \} = 0$

$$\Rightarrow k^2 - n^2 = 0 \text{ , i.e. } \underline{k = \pm n}$$

For $\lambda = 1$ we obtain $a_1 \{ (k+1)\cancel{k} + k + 1 - n^2 \} = 0$

$$\Rightarrow a_1 \{ k+1-n \} \{ k+1+n \} = 0$$

$$\Rightarrow a_1 = 0 \quad (\text{except for } k = \pm n = -\tfrac{1}{2} \;)$$

Recursion relation $(k = n)$:

$$a_j \{ (n+j)(n+j-1) + (n+j) - n^2 \} + a_{j-2} = 0$$

renumber $j \to j+2$ $\Rightarrow$ $a_{j+2} = - a_j \dfrac{1}{(j+2)(2n+j+2)}$

$$a_2 = - a_0 \frac{1}{2(2n+2)} = - a_0 \frac{n!}{2^2 \, 1! \, (n+1)!}$$

$$a_4 = - a_2 \frac{1}{4(2n+4)} = a_0 \frac{n!}{2^4 \, 2! \, (n+2)!}$$

$$a_6 = - a_4 \frac{1}{6(2n+6)} = - a_0 \frac{n!}{2^6 \, 3! \, (n+3)!}$$

$$\Rightarrow a_{2p} = (-1)^p \frac{n!}{2^{2p} \, p! \, (n+p)!} a_0$$

and $y_n(x) = a_0 \sum_{j=0}^{\infty} (-1)^j \dfrac{n! \, x^{n+2j}}{2^{2j} \, j! \, (n+j)!} = a_0 2^n n! \underbrace{\sum_{j=0}^{\infty} (-1)^j \frac{1}{j! (n+j)!} \left(\frac{x}{2}\right)^{n+2j}}_{J_n(x)}$

The parity is even or odd as determined by n

From the equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$

$x \to -x$ $\quad (-x)^2 \dfrac{d^2 y}{d(-x)^2} + (-x) \dfrac{dy}{d(-x)} + ((-x)^2 - n^2) y = 0$

~~the parity is~~

If the second root to the indicial equation $k=-n$ is not an integer we obtain the second solution directly, but for integer n there will be a value j for which the resulting coefficient $\frac{1}{(j+2)(-2n+j+2)}$ diverges. The solution in that case is actually

$$J_{-n}(x) = (-1)^n J_n(x)$$, i.e. the same solution as before.

The linearly independent solution $N_n(x)$ (Neumann function) in this case needs to be constructed using different techniques.

Regular and irregular singularities

$y'' - \frac{6}{x^2} y = 0$; regular singularity at $x=0$
indicial equation $s(s-1)-6=0 \Rightarrow s = \begin{cases} 3 \\ -2 \end{cases}$
No recursion relation, $y_1 = x^3$, $y_2 = x^{-2}$

$y'' - \frac{6}{x^3} y = 0$; irregular singular point at $x=0$
indicial equation (for x^{s-3})
$-6a_0 = 0 \Rightarrow$ No solution for series ansatz

$y'' + \frac{1}{x} y' - \frac{b^2}{x^2} y = 0$; regular singularity at $x=0$
indicial eq. $s(s-1)+s-b^2=0 \Rightarrow s = \pm b$
Homogeneous in $x \Rightarrow$ no recurrence relation but $y_1 = x^b$, $y_2 = x^{-b}$

$y'' + \frac{1}{x^2} y' - \frac{b^2}{x^2} y = 0$; irregular singularity at $x=0$
indicial eq. (from y' term) $s=0$

$$\sum_{j=0}^{\infty} a_j (j+s)(j+s-1) x^{j+s-2} + \sum_{j=0}^{\infty} a_j (j+s) x^{j+s-3} - b^2 \sum_{j=0}^{\infty} a_j x^{j+s-2} = 0$$

recursion $a_{j+1} = \frac{b^2 - j(j-1)}{j+1} a_j$

unless $b^2 = n(n-1)$ for some n (integer) making the series terminate $\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim \frac{j(j+1)}{j+1} \to \infty$

Divergence for all $x \neq 0$

Fuchs' theorem

At least one power-series solution can always be obtained provided that we expand about an ordinary point or at worst a regular singular point.
If the expansion is attempted about an irregular or essential singularity the method may fail.

- If the two roots of the indicial equation are equal only one solution will be found using series expansion
- If the two roots differ by a nonintegral number, two independent solutions may be obtained
- If the two roots differ by an integer, the larger of the two will yield a solution, while the smaller may or may not give a solution, depending on the behavior of the coefficients.

Rate of convergence: Computer applications

Second solution:

Linear independence $\{\varphi_k(x)\}_{k=1}^n$ linearly independent if $\sum_{k=1}^n a_k \varphi_k(x) = 0$ requires $a_k = 0$ for all k.

Assume φ_k differentiable and generate n linear equations

$$\sum_{k=1}^n a_k \varphi_k'(x) = 0$$

$$\sum_k a_k \varphi_k''(x) = 0$$

$$\vdots$$

$$\sum_k a_k \varphi_k^{(n-1)}(x) = 0$$

$$\begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix} \neq 0 \Rightarrow a_k = 0$$

only solution

Wronskian

$\{\varphi_k\}$ analytical

$W\{\varphi_j\} = 0$ at isolated values of the argument does not prove

ODE $y'' + P(x)y' + Q(x)y = 0$

Let y_1 and y_2 be two independent solutions

The Wronskian $W = y_1 y_2' - y_1' y_2$

Differentiate: $W' = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' =$

$$= y_1(-P y_2' - Q y_2) - (-P y_1' - Q y_1) y_2 =$$
$$= -P(y_1 y_2' - y_1' y_2) = -PW$$

Separate and integrate $\frac{dW}{W} = -P dx$

$$\ln \frac{W(x)}{W(a)} = -\int_a^x P(x_1) dx_1 \iff W(x) = W(a) \exp\left[-\int_a^x P(x$$

But $W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) \Rightarrow$

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a) \frac{\exp\left[-\int_a^x P(x_1) dx_1\right]}{y_1^2}$$

integrate: $$y_2(x) = y_1(x) W(a) \int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1) dx_1\right]}{[y_1(x_2)]^2} dx_2$$

term $\frac{y_2(b)}{y_1(b)} y_1(x)$ dropped since it is a multiple of y_1 (known solution

Similarly the lower bounds either scale the solution ($x_1 = a$) or add a multiple of the first ($x_2 = b$)

Drop lower bounds and set $W(a) = 1$ to obtain

$$\boxed{\begin{gathered} y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} P(x_1) dx_1\right]}{[y_1(x_2)]^2} dx_2 \\ \text{Special case } P(x) = 0 \\ y_2(x) = y_1(x) \int^x \frac{dx_2}{[y_1(x_2)]^2} \end{gathered}}$$

Second solution generated from the first

Series form of the second solution

$$y'' + P(x)y' + Q(x)y = 0$$

Three steps:

1) Power series expansion of $P(x) = \sum_{i=-1}^{\infty} p_i x^i$ and $Q(x) = \sum_{j=-2}^{\infty} q_j x^j$ (satisfying Fuchs' theorem giving at most a regular singularity). Of course $P(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (General case).
2) Develop first few terms of series for y_1
3) Use this solution to obtain y_2 integrating term wise

$$y'' + (p_{-1}x^{-1} + p_0 + p_1 x + \dots)y' + (q_{-2}x^{-2} + q_{-1}x^{-1} + q_0 + q_1 x + \dots)y = 0$$

if $p_{-1} = q_{-2} = q_{-1} = 0$ then $x = 0$ is an ordinary point

The ansatz $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{s+\lambda}$ gives

$$\sum_{\lambda=0}^{\infty}(s+\lambda)(s+\lambda-1)a_\lambda x^{\lambda+s-2} + \sum_{i=-1}^{\infty} p_i x^i \sum_{\lambda=0}^{\infty}(s+\lambda)a_\lambda x^{s+\lambda-1} +$$

$$+ \sum_{j=-2}^{\infty} q_j x^j \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+s} = 0$$

Lowest power $(p_{-1} \neq 0, q_{-2} \neq 0)$: x^{s-2}

Indicial equation: $\{s(s-1) + p_{-1}s + q_{-2}\}a_0 = 0$

$a_0 \neq 0 \Rightarrow s^2 + (p_{-1} - 1)s + q_{-2} = 0$

Two roots with $s_1 = \alpha$ and $s_2 = \alpha - n$ where $n = 0$ or positive integer making s_1 the larger root. If n not an integer we can obtain the second solution as a power series and we're done – consider case where n is an integer.

Thus, $(s-\alpha)(s-\alpha+n) = 0$

$$\Rightarrow s^2 + (n-2\alpha)s + \alpha(\alpha-n) = 0$$

and $s^2 + (p_{-1} - 1)s + q_{-2} = 0 \Rightarrow p_{-1} - 1 = n - 2\alpha$

For $s = \alpha$ we may write $y_1(x) = x^\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda$

We can directly construct $y_2(x)$ as

$$y_2(x) = y_1(x) \int^{x} \left[\frac{\exp\left(-\int^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1\right)}{x_2^{2\alpha} \left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2} \right] dx_2$$

(lower bound in integration over x_1 only gives constant factor)

The exponential $\int^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1 = p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}$

gives

$$\exp\left[-\int^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i dx_1\right] = x_2^{-p_{-1}} \cdot \exp\left[-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right] =$$

$$= x_2^{-p_{-1}} \cdot \left[1 - \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + \frac{1}{2!}\left(-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right)^2 + \ldots\right]$$

Taylor expansion of the exponential convergent if the expansion of $P(x)$ uniformly convergent

The denominator

$$\left[x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2\right]^{-1} = x_2^{-2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^{-2} = x_2^{-2\alpha} \sum_{\lambda=0}^{\infty} b_\lambda x_2^\lambda$$

different

and $$y_2(x) = y_1(x) \int^{x} x_2^{-p_{-1}-2\alpha} \left(\sum_{\lambda=0}^{\infty} c_\lambda x_2^\lambda\right) dx_2$$

but $-p_{-1} - 2\alpha = -n-1$

$$\Rightarrow y_2(x) = y_1(x) \int^{x} \left(c_0 x_2^{-n-1} + c_1 x_2^{-n} + c_2 x_2^{-n+1} + \ldots + c_n x_2^{-1} + \ldots\right) dx_2$$

The coefficient of $y_1(x)$ consists of

1) Power series starting with x^{-n}
2) A logarithm term from integration of $\frac{c_n}{x_2}$ unless $c_n = 0$. Always appears when n is integer unless $c_n = 0$

$$\therefore \quad y_2(x) = y_1(x)\left\{\ln|x| + \sum_{j=-n}^{\infty} d_j x^j\right\}$$

different

The second solution usually diverges at the origin due to the logarithm and negative powers

$\Rightarrow$ <u>irregular solution</u>

Inhomogeneous linear ODE's

$$y'' + P(x)y' + Q(x)y = F(x)$$

Assume $y_1(x)$ and $y_2(x)$ linearly independent solutions of the homogeneous equation

A general form of the particular solution can be written
$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ in terms of the known y_1 and y_2

Take the derivatives

$$y_p' = u_1 y_1' + u_2 y_2' + (y_1 u_1' + y_2 u_2')$$

Restrict to $y_1 u_1' + y_2 u_2' = 0$ (If we find unique solution then this is ok)

$$\Rightarrow y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Put in ODE: $(u_1'y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'') + P(x)(u_1 y_1' + u_2 y_2') +$
$+ Q(x)(u_1 y_1 + u_2 y_2) = F(x)$

But $u_1\{y_1'' + P(x)y_1' + Q(x)y_1\} = 0$ and same for y_2

$$\Rightarrow u_1' y_1' + u_2' y_2' = F(x)$$

We have two equations

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = F(x) \end{cases} \iff \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F(x) \end{pmatrix}$$

Unique solution if $\mathrm{Det}\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \equiv W\{y_1, y_2\} \neq 0$
but y_1 and y_2 are linearly independent

Solve for u_1' and u_2' : $\quad u_1' = -\frac{y_2}{y_1} u_2'$

Put into second equation : $\quad y_1'\left(-\frac{y_2}{y_1}u_2'\right) + y_2' u_2' = F$

$$u_2'\left\{ y_2' - \frac{y_2 y_1'}{y_1} \right\} = F$$

$$u_2(x) = \int^x \frac{y_1(s)F(s)}{W\{y_1,y_2\}}\,ds \longleftarrow u_2' = \frac{y_1(x)\,F(x)}{y_1 y_2' - y_1' y_2} = \frac{y_1(x)F(x)}{W\{y_1,y_2\}}$$

and $\quad u_1' = -\frac{y_2}{y_1}\frac{y_1\,F}{W\{y_1,y_2\}} = -\frac{y_2(x)\,F(x)}{W\{y_1,y_2\}}$

$$u_1(x) = -\int^x \frac{y_2(s)\,F(s)}{W\{y_1,y_2\}}\,ds$$

$$\therefore \quad y_p(x) = y_2(x)\int^x \frac{y_1(s)F(s)}{W\{y_1,y_2\}}\,ds - y_1(x)\int^x \frac{y_2(s)F(s)}{W\{y_1,y_2\}}\,ds$$

and $\quad y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$

$$(1-x)y'' + xy' - y = (1-x)^2$$

Put on normal form $\quad y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 1-x$

Two solutions $y_1 = x$ and $y_2 = e^x$ of homogeneous eq.

Solve for u_1' and u_2' : $\begin{cases} x u_1' + e^x u_2' = 0 & (i) \\ u_1' + e^x u_2' = 1-x & (ii) \end{cases}$

from (i) $\quad u_1' = -\frac{e^x}{x}u_2' \quad$ into (ii) $\quad u_2'\left(e^x - \frac{e^x}{x}\right) = 1-x$

$$u_2'\left(\frac{x-1}{x}\right)e^x = 1-x \implies u_2' = -xe^{-x}$$

$$u_1' = 1$$

$\therefore \quad u_1 = x, \quad u_2 = (x+1)e^{-x}$

$$y_p(x) = u_1 y_1 + u_2 y_2 = x \cdot x + (x+1)e^{-x}\cdot e^x = x^2 + x + 1$$

$$= x^2 + 1 \quad (y_1 = x)$$

$$y(x) = C_1 x + C_2 e^x + x^2 + 1$$

Using the formula: $W\{x, e^x\} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1)$

$$\int^x \frac{y_1(s)F(s)}{W\{y_1,y_2\}}ds = \int^x \frac{s(1-s)}{e^s(s-1)}ds = -\int^x s e^{-s} ds = (x+1)e^{-x}$$

$$\int^x \frac{y_2(s)F(s)}{W\{y_1,y_2\}}ds = \int^x \frac{e^s(1-s)}{e^s(s-1)}ds = -\int^x ds = -x$$

$$y_p(x) = (x+1)e^{-x} \cdot e^x - x(-x) = x + 1 + x^2$$

Simpler to just solve the linear equations than to remember formula.

x ——— Non linear Differential Equations ——— x

Bernoulli equation $\quad y'(x) = p(x)y(x) + q(x)[y(x)]^n$

$p(x), q(x)$ real and $n \neq 0, 1$

substitute $u(x) = [y(x)]^{1-n}$ then

$$u' = (1-n)y^{-n} \cdot y' = (1-n)y^{-n}\{p(x)y + q(x)y^n\} =$$
$$= (1-n)\{p(x)y^{1-n} + q(x)\} =$$
$$= (1-n)\{p(x)u + q(x)\}$$ which can be solved using integrating factor

Riccati equation: $\quad y' = p(x)y^2 + q(x)y + r(x)$, $p \neq 0$, $r \neq 0$

No known general method but if one special solution (guessed or inspection) $y_0(x)$ is known then

$y = y_0 + u$ gives the general solution with

u satisfying $\quad u' = pu^2 + (2py_0 + q)u \quad$ Bernoulli e

Insert $y = y_0 + u$: $\cancel{y_0'} + u' = p(x)(y_0+u)^2 + q(x)(y_0+u) + r(x) =$

$$= \cancel{p(x)y_0^2} + (2p(x)y_0 + q(x))u + \cancel{q(x)y_0} + \cancel{r(x)}$$