Mathematical Methods for Physicists, FK7048, fall 2019
Exercise sheet 8, **due monday october 28th, 10:00**

Total amount of points: 6p.

1 The Basel problem

The purpose of this exercise is to calculate the sum of the following (converging) series by different methods:

$$S = \sum_{n=1}^{+\infty} \frac{1}{n^2} \tag{1}$$

(a) (1p) As seen in class, apply the residue theorem toan appropriate function with an appropriate contour.

You can construct a 2π-periodic function f over $\mathbb{R}$ by defining it on a period, for instance the interval $]-\pi,\pi]$. Then you are allowed to expand f in Fourier series.

(b) (1p) Take $f(x) = x$ on $]-\pi,\pi]$ and apply Parseval's theorem to find S.

(c) (1p) Take $f(x) = x^2$ on $]-\pi,\pi]$ and apply Dirichlet's theorem in a particular point to get S.

2 Retarded potentials

Let $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ be the D'Alembertian in 3D, $V(\boldsymbol{r},t)$ the electric potential, $\boldsymbol{A}(\boldsymbol{r},t)$ the magnetic vector potential, $\rho(\boldsymbol{r},t)$ the charge density and $\boldsymbol{j}(\boldsymbol{r},t)$ the electric current density. One can show that Maxwell's equations in the Lorenz[1]gauge are equivalent to wave equations with source terms:

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{2}$$

$$\Box\ \boldsymbol{A} = \mu_0 \boldsymbol{j} \tag{3}$$

$$\Box\ V = \frac{\rho}{\epsilon_0} \tag{4}$$

(a) (2p) Calculate the retarded Green's function $G(\boldsymbol{r},t;\boldsymbol{r'},t')$ of the D'Alembertian using a space-time Fourier transform defined as follows for functions[2] $f \in L^2(\mathbb{R}^4)$:

$$\widehat{f}(\boldsymbol{k},\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} f(\boldsymbol{r},t)\ e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\ d^3r\ dt \tag{5}$$

$$f(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \widehat{f}(\boldsymbol{k},\omega)\ e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\ d^3k\ d\omega \tag{6}$$

Hints: - Actually $G(\boldsymbol{r},t;\boldsymbol{r'},t') = G(\boldsymbol{r}-\boldsymbol{r'},t-t')$, moreover one can take $\boldsymbol{r'} = \boldsymbol{0}$ and $t' = 0$ during the calculation. Justify this.
- When inverting the Fourier transform in time, choose your contour so that $G(\boldsymbol{r},t) = 0$ for $t \leq 0$. Justify this choice.

[1]Note that the name is spelled correctly, this gauge is due to Lorenz, the transformations are due to Lorentz.

[2]Fourier transformation is an isomorphism only on Schwartz spaces and by a subtle extension to L^2 spaces, however the Fourier inversion formula also holds if the function and its Fourier transform both belong to an L^1 space. In a nutshell, be always careful when inverting a Fourier transform!

(b) (1p) Infer the so-called retarded potentials:

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{\rho\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r} - \boldsymbol{r'}|} \, d^3r' \tag{7}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{j}\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r} - \boldsymbol{r'}|} \, d^3r' \tag{8}$$