Mathematical Methods for Physicists, FK7048, fall 2018
Exercise sheet 7, **Submit exercise 7 and exercise 8 before 25^{th} October**

Total amount of points: 13 p + 3 bp.

1 Integration of Bessel functions

Let us define the following sequence for $n \in \mathbb{N}$:

$$I_n = \int_0^{+\infty} J_n(x)dx \tag{1}$$

where the $J_n(x)$ are Bessel functions of the first kind.

(a) (0.5p) Give the asymptotic expression of $J_n(x)$ at $x = 0$ and at infinity. You don't need to derive them, simply refer to a formula of the book.

(b) (1p) Using a recurrence relation (don't prove it, simply give the reference) and the result of the previous question, to show that:

$$I_1 = -\left[J_0(x)\right]_0^{+\infty} = 1 \tag{2}$$

(c) (0.5p) Thanks to a recursion relation (don't prove it, simply give the reference), show that $\forall n \in \mathbb{N}^\star,\ I_{n-1} = I_{n+1}$.

(d) (1.5p) Compute I_0 and conclude.

Hint: Use the complex integral representation of J_0 and a proper integral form of the Dirac distribution. If you use a particular theorem in your calculation, you should state this explicitly.

2 Plateau-Rayleigh instability

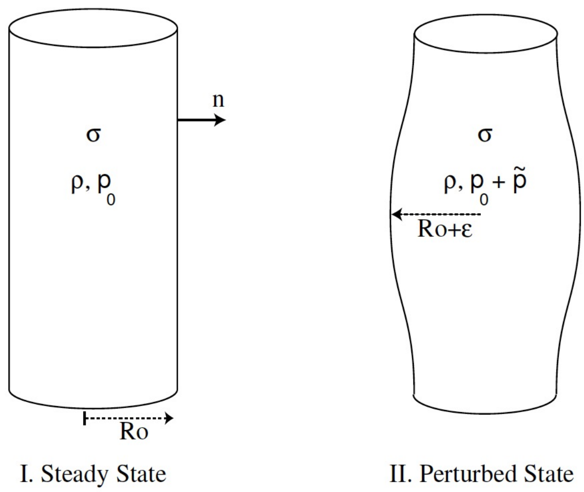

Figure 1: Perturbation of a cylindrical column of fluid

The principle of an instability is simple: one has an equilibrium state, one perturbs it and wants to study the evolution of this perturbation. Assuming the perturbation is small, one can linearise the equations describing the dynamics and look for solutions proportional to $e^{i(kx-\omega t)}$ where $k \in \mathbb{R}$ is the wave number and $\omega \in \mathbb{C}$ is the angular frequency. The result of the analysis is a function $\omega(k)$ called dispersion relation. If ω is real, one has a travelling mode. If ω is imaginary, one has a growing or decaying mode.

Consider an infinitely long cylindrical column of fluid of density ρ at equilibrium with radius R_0. We neglect gravity so according to hydrostatics equation (cf. sheet 1, ex 1), the pressure is constant in the fluid. Let us denote it p_0 and assume that the atmospheric pressure is zero. Then the Young-Laplace law yields $p_0 = \frac{\sigma}{R_0}$ where σ is the surface tension. One perturbs the surface of the cylinder which now has a radius:

$$\mathcal{R}(z,t) = R_0 + \epsilon \; e^{i(kz-\omega t)} \quad \text{with} \quad \epsilon \ll R_0 \; . \tag{3}$$

The pressure in the fluid is now given by $p(r,z,t) = p_0 + \widetilde{p}(r,z,t)$ and there is a velocity field $\widetilde{\boldsymbol{u}}(r,z,t)$ describing the velocity of the perturbation.

The cylindrical symmetry is preserved so that these fields do not depend on the azimuth θ. The linearised Euler and continuity equations in cylindrical coordinates are:

$$\frac{\partial \widetilde{u}_r}{\partial t} = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial r} \tag{4}$$

$$\frac{\partial \widetilde{u}_z}{\partial t} = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial z} \tag{5}$$

$$\frac{\partial \widetilde{u}_r}{\partial r} + \frac{\widetilde{u}_r}{r} + \frac{\partial \widetilde{u}_z}{\partial z} = 0 \tag{6}$$

We are interested in the stability of a Fourier mode k.

(a) (0.5p) Look for solutions of the form:

$$\widetilde{u}_r(r,z,t) = R(r) \; e^{i(kz-\omega t)} \tag{7}$$

$$\widetilde{u}_z(r,z,t) = Z(r) \; e^{i(kz-\omega t)} \tag{8}$$

$$\widetilde{p}(r,z,t) = P(r) \; e^{i(kz-\omega t)} \tag{9}$$

and rewrite the equations in terms of $R(r)$, $Z(r)$, and $P(r)$.

(b) (1p) Eliminate $Z(r)$ and $P(r)$ in order to obtain the following 2nd order linear ODE for $R(r)$:

$$r^2 \; \frac{d^2R}{dr^2} + r \; \frac{dR}{dr} - \left(1 + k^2r^2\right)R(r) = 0 \tag{10}$$

(c) (1p) Equation 10 has got striking similarity with the bessel differential equation and is knows as **modified bessel equatoin**. It differs from the original bessel equation only in the sign of k^2r^2, but this small change is sufficient to alter the nature of the solution. The solution of this equation is not oscilatory but exponential in nature.
Further, it can also be noticed that we can get solution to this equation by substituting $k \rightarrow ik$, which shows that if $P(kr)$ is a solution to the bessel ODE, then $P(ikr)$ must be a solution of the modified bessel equation. So in general, we can define two linearly independent solution as $I_\nu(ikr)$ and $K_\nu(ikr)$ for the equation given below:

$$r^2 \; \frac{d^2R}{dr^2} + r \; \frac{dR}{dr} - \left(\nu + k^2r^2\right)R(r) = 0 \tag{11}$$

Where $I_\nu(x) = i^{-\nu}J_\nu(ix)$ is the solution of first kind. The prefactor of $i^{-\nu}$ should be noticed to make the solution real for $x \in \mathbb{R}$. Though the second solution $K_\nu(x)$ can be described as shown below for a non-integer ν:

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1}[J_\nu(ix) + iY_\nu(ix)] \tag{12}$$

If ν is integer the equation 12 is not defined, but then the second solution can be defined as $K_\nu(x) = \lim_{\alpha\rightarrow\nu} K_\alpha(x)$.

Given the information above and for an integer $\nu = n$, first write down the series expression of $I_n(x)$ and also show that

$$K_n(x) = \lim_{\nu\rightarrow n} \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{sin(\nu\pi)}$$

(d) (1p) Now let's try to get the recurrence relation for the problem. For bessel functions we already have

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \tag{13}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x) \tag{14}$$

Get the similar relation for $I_\nu(x)$ also.

(e) (1.5p) With the expression stated above, write down the general solution of equation 10. Show that one of the two linearly independent solutions diverges at $r = 0$; remove it so that your solution has the form $R(r) = C\ f(r)$, where the constant $C \in \mathbb{C}$ will be determined below. Using the property of the special function f, infer $P(r)$. *(Hint: Use L'Hospital's rule to calculate the limtis if you need to.)*

We now the apply boundary conditions.

(f) (1p) The kinematic condition at the fluid/air interface is $\frac{\partial \mathcal{R}}{\partial t} = \widetilde{u}_r(R_0, z, t),\ \forall (z,t) \in \mathbb{R} \times \mathbb{R}^+$. Use it to infer the constant C.

(g) *Ultra bonus question:* (2p) The dynamic condition is given by the general form of Young-Laplace law: $p(\mathcal{R}, z, t) = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ where R_1 and R_2 are the principal radii of curvature of the deformed cylinder. Justify that:

$$p(R_0, z, t) = \frac{\sigma}{R_0} - \frac{\epsilon\sigma}{R_0^2}\left(1 - k^2 R_0^2\right) e^{i(kz-\omega t)} \quad \text{for} \quad \epsilon \ll R_0 \tag{15}$$

(h) (1p) Combining (15) with your result of question (d), find the dispersion relation:

$$\omega^2 = \frac{k\sigma}{\rho R_0^2}\ g(kR_0)\left(k^2 R_0^2 - 1\right) \quad \text{where } g \text{ is a ratio of special functions.} \tag{16}$$

(i) (1p) *Bonus question:* Discuss the stability of Fourier modes. When do you observe waves? When and how does the perturbation grow? Plot the growth rate $|\Im(\omega)|$ as a function of k. Note that strictly speaking, we chose the sign of the imaginary part of ω when we took the square root, so we assumed that the amplitude of the mode does not decay, but grows.

3 Playing with Legendre polynomials

Let P_n be the Legendre polynomials for $n \in \mathbb{N}$.

(a) (1p) We define $Q(x) = 10x^3 - 3x^2 - 6x + 1$ and the sequence:

$$\forall n \in \mathbb{N}, \quad I_n = \int_{-1}^{1} Q(x) P_n(x) dx \tag{17}$$

Show that $I_n = 0,\ \forall n \in \mathbb{N}\backslash\{2,3\}$ and calculate I_2 and I_3.

(b) (1.5p) Show the following relation :

$$\int_{-1}^{1} x^n P_n(x) dx = \frac{2n!}{(2n+1)!!} \tag{18}$$

Hint: Integrate Rodrigues' formula by parts until you recognize a beta function.