Mathematical Methods for Physicists, FK7048, fall 2019
Exercise sheet 6, **due Wed october 16th**

Total amount of points: 13 p + 2 bp.

1 Green's functions

In this problem, f is an arbitrary real-valued continuous function, unless stated otherwise.

(a) (1p) Consider the following 2nd order inhomogenous ODE with constant coefficients:

$$a\, y''(x) + b\, y'(x) + c\, y(x) = f(x) \quad (a, b, c) \in \mathbb{C}^3 \tag{1}$$

This equation is not self-adjoint. Show that the Green's function associated to (1) and the boundary conditions $y(x_1) = y(x_2) = 0$ still satisfies some continuity and discontinuity conditions. Which main property that Green's functions for adjoint problems satisfy is lost? Which condition on (1) should you impose to have this property?

(b) (1p) Construct the Green's function associated to the following initial value problem:

$$\begin{cases} y''(x) + 2y'(x) + y(x) = f(x) \\ y(0) = 0 \quad \text{and} \quad y'(0) = 0 \end{cases} \tag{2}$$

Assume that the continuity and discontinuity conditions established in the previous question also hold for this kind of homogeneous boundary conditions. Give the final expression for $y(x)$ in terms of f.

(c) (2p) Construct the Green's function associated to the following boundary value problem:

$$\begin{cases} y''(x) + y'(x) - 2y(x) = f(x) \\ y(0) = 1 \quad \text{and} \quad \lim_{x \to +\infty} y(x) = 2 \end{cases} \tag{3}$$

Find an appropriate function g such that $u(x) = y(x) - g(x)$ satisfies homogeneous boundary conditions and perform this change of dependent variable in the ODE. Give the final expression of $y(x)$ in terms of f and g.

Hint : The main difficulty comes from the boundary at infinity. Try to find a function which has a finite value in $x = 0$ and vanishes at infinity, then use it to construct g ; choose it such that the RHS of the ODE changes the least possible.

Show that this boundary value problem has no solution when $f = 0$. However, the Green's function method still provides a solution: why is it wrong?

2 Complex integrals with a branch cut

(a) (1p) Is the complex logarithm continuous in the complex plane? The same question for the function $z \to z^\alpha,\ \alpha \in [0, +\infty]$.

(b) Consider the following integral:

$$I = \int_0^{+\infty} \frac{dx}{x^\alpha (x + p)} \tag{4}$$

(b1) (0.5p) For which values of α does this integral converge?

(b2) (2p) Using the contour of the figure 11.26 from the book, show that $I = \Gamma(\alpha)\Gamma(1 - \alpha)p^{-\alpha}$.

(c) Consider the following integral:

$$J = \int_0^{+\infty} \frac{\log(x)}{a^2 + x^2} \quad \text{for} \quad a \in [0, +\infty[\tag{5}$$

(c1) (0.5p) What is $\text{Log}(x)$ for $x \in]-\infty, 0[$ where Log is principal determination of the complex logarithm?

(c2) (2p) Use a contour in the upper complex plane (exclusively) dealing with the branch cut of Log to compute J. What happens when $a = 0$? Can you show this without using your result for the integral?

3 Gamma and Zeta functions in physics

(a) (1p) Compute the moments of the Bose-Einstein distribution for $n \in \mathbb{N}^\star$:

$$\int_0^{+\infty} \frac{x^n}{e^x - 1}\, dx = n!\, \zeta(n+1) \tag{6}$$

Hint: Use the following series expansion for a suitable X:

$$\frac{1}{1-X} = \sum_{p=0}^{+\infty} X^p \quad \text{provided that} |X| < 1 \tag{7}$$

(b) (1p) Infer the moments of the Fermi-Dirac distribution for $n \in \mathbb{N}^\star$:

$$\int_0^{+\infty} \frac{x^n}{e^x + 1}\, dx = \left(1 - \frac{1}{2^n}\right) n!\, \zeta(n+1) \tag{8}$$

Hint: One way to solve the this problem is to note that $e^x - 1 = \left(e^{\frac{x}{2}}\right)^2 - 1$.

4 Bonus: the volume of hyperballs

In physics, we often need to compute N-dimensional integrals with spherical symmetry. For instance let f be a real-valued integrable function, then:

$$\int_{\mathbb{R}^N} f\left(\sqrt{\sum_{i=1}^N x_i^2}\right) \prod_{i=1}^N dx_i \equiv \int_0^{+\infty} f(r)\, d\big(a_N r^N\big) \tag{9}$$

where $a_N r^N$ is the hypervolume of the N-dimensional hyperball. We have simply passed from Cartesian to spherical coordinates and we wish to calculate the coefficient a_N. To do that, a nice idea is to choose f as the Gaussian i.e. $f(r) = e^{-r^2}$.

(a) (0.5 bp) For this choice of f, you can factorize the left-hand side of (9). Show it is equal to $\pi^{\frac{N}{2}}$.

(b) (1 bp) By a straighforward change of variables, show that the right-hand side of (9) is equal to $a_N\, \Gamma\big(\frac{N}{2} + 1\big)$ for this choice of f.

(c) (0.5 bp) Deduce the value of a_N and verify that a_1, a_2 and a_3 take the familiar values. What is the geometrical meaning of $d\big(a_N r^N\big)$?