Mathematical Methods for Physicists, FK7048, fall 2019
Exercise sheet 5, **due Wednesday october 9th**

Total amount of points: 12 p + 2 bp.

1 Residue calculus

(a) (0.5p) Compute the residue in $z = 0$ of $f(z) = z^{10}\, e^{\frac{1}{z^2}}$ and state the type of singularity.

(b) (1p) Compute the residues in $z = 0$ and $z = 2$ of $h(z) = \frac{\pi \cot(\pi z)}{z-2}$ and state the type of singularity for both points. Mittag-Leffler's theorem might be useful but you can also use another method.

2 Integral calculus

(a) (0.5p) State the two Jordan lemmas. You can refer to them later on, but you will have to detail their application.

(b) (1p) Change the following real integral into a complex one in order to compute it:

$$I = \int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta) + \sin(\theta)} \tag{1}$$

(c) (1p) Let $D_a \cong \{t + ia, t \in \mathbb{R}\}$ be the real line shifted by $a > 0$ in the upper complex plane. Show that the integral of the Gaussian e^{-z^2} over D_a is equal to its integral over the real line and compute its value explicitly:

$$\int_{-\infty}^{+\infty} e^{-(x+ia)^2} dx \equiv \int_{-\infty}^{+\infty} e^{-x^2} dx = \,? \tag{2}$$

Hint: Apply Cauchy's theorem to the rectangle $\{(-R, 0), (R, 0), (R, a), (-R, a)\}$ and then take the limit R goes to infinity.

(d) (1p) By a principal value calculation similar to the one done for the Dirichlet integral, compute the following integral:

$$J = \int_0^{+\infty} \frac{\sin(x)}{x(x^2 + a^2)}\, dx \quad \text{for} \quad a \in [0, +\infty[\tag{3}$$

What happens when $a = 0$? Can you show this without using the explict result for the integral?

3 Primitive calculus

In the book, the following integral is computed thanks a tricky contour.

$$K = \int_0^{+\infty} \frac{dx}{x^3 + 1} \tag{4}$$

The integrand is a rationnal function so it is actually possible to determine its primitive.

(a) (1p) Factorize x^3+1; feel free to use Horner algorithm if you know it. Then decompose $f(x) = \frac{1}{x^3+1}$ in real partial fractions.

(b) (2p) Obtain a primitive of $f(x)$:

$$F(x) = \int^x f(t)dt = \frac{1}{3}\ln(1 + x) - \frac{1}{6}\ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x - 1}{\sqrt{3}}\right) \tag{5}$$

(c) (1p) Find back the result given in the book by computing limits of $F(x)$, namely $K = \frac{2\pi}{3\sqrt{3}}$.

(d) *Bonus*: (1p) calculate $\int_0^{+\infty} \frac{dx}{x^p+1}$, where p is an integer $p \geq 2$.

4 Kepler's orbits

Consider two point particles with masses m_1 and m_2, in gravitational interaction. In the course analytical mechanics, you will learn that one can derive the equations of motion from a Lagrangian, $\mathscr{L} = T - U$, there T and U are the kinetic and potential energy. The Lagrangian of the current system is:

$$\mathscr{L} = \frac{1}{2}m_1\dot{\boldsymbol{r}}_{\boldsymbol{1}}^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_{\boldsymbol{2}}^2 + G\frac{m_1 m_2}{|\boldsymbol{r_1} - \boldsymbol{r_2}|} \tag{6}$$

where $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are the position vectors of the particles, G is the gravitational constant and "$\dot{}$" means the time derivative. It is one of the very few systems for which the equations of motion can be analytically solved.

(a) *Bonus*: (1p) In order to decouple the motion of the centre of mass and the relative motion of the particles, perform the following coordinates transformation:

$$\begin{cases} (m_1 + m_2)\boldsymbol{R} = m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} \\ \boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2} \end{cases} \tag{7}$$

Show that the Lagrangian can be written in terms of the new variables as follows:

$$\mathscr{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + \frac{K}{r} \tag{8}$$

with $M = m_1 + m_2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$, $K = GM\mu$ and $r = |\boldsymbol{r}|$.

(b) (1p) We look for solutions that lie in a plane. Therefor, use the cartesian coordinates $\boldsymbol{R} = X\ \boldsymbol{e}_x + Y\ \boldsymbol{e}_y$ for the center of mass, and polar coordinates $\boldsymbol{r} = r\ \boldsymbol{e}_r(\theta)$ for the relative motion. We view the relative distance as a function of θ.

In deriving the Euler-Lagrange equations, that is, the equations of motion, we view a coordinate and its time derivative as independent, so we write $\mathscr{L} = \mathscr{L}(q, \dot{q})$, where $(q, \dot{q})$ stands for any of the coordinates, $(X, \dot{X})$, $(r, \dot{r})$ etc. The equations of motion are given by the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} = \frac{\partial \mathscr{L}}{\partial q}\ . \tag{9}$$

Obtain the equation of motion for each coordinate q from the Euler-Lagrange equations. For r, you should obtain the following 2nd order non-linear ODE:

$$\mu\ \ddot{r} = \frac{l^2}{\mu r^3} - \frac{K}{r^2} \quad \text{with} \quad l := \mu r^2\dot{\theta} \equiv \text{constant.} \tag{10}$$

What is the motion of the center of mass? What is the physical interpretation of l?

(c) (1p) Perform the transformation $u = \frac{1}{r}$ with u seen as a function of θ (instead of time) and apply the chain rule to the differential operator: $\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}$. The expressions for the velocity and acceleration in terms of u are known as Binet's formulae.

(d) (1p) Solve the equation for $u(\theta)$ and conclude on the possible orbits in the center of mass frame.