Mathematical Methods for Physicists, FK7048, fall 2019
Exercise sheet 2, **due Monday september 23rd, 10:00**

Total amount of points: 11 p, and 1 bonus points (2 bp).

1 Laplace equation

Let us consider an incompressible inviscid flow around a sphere of radius R. Far away from the sphere, the fluid has a constant velocity $\vec{u} = U\ \vec{e}_z$. The problem is independent of the azimuthal angle φ because of the symmetry of the incoming flow around the z-axis. Therefore, it is sufficient to look at the flow in the (x, z) plane, by setting $\varphi = 0$ and choosing the range of the polar angle as $0 \leq \theta < 2\pi$.

We assume that the flow is irrotational so that there exists a potential function $\phi(x, z)$ such that $\vec{u} = \vec{\nabla}\ \phi$. Because $\vec{\nabla} \cdot \vec{u} = 0$ due to the incompressibility, we find that ϕ is a solution of the Laplace equation in spherical coordinates (for the current problem, we drop the last term):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0\ . \tag{1}$$

The purpose of this exercise is to solve this equation by the method of multipole expansion. Due to the absence of viscosity, one imposes a perfect slipping condition at the surface of the sphere:

$$u_r(R, \theta) = 0, \forall \theta \in [0, 2\pi[\tag{2}$$

In electrostatics, one solves the Poisson equation to find the electric potential,

$$\Delta\varphi = -\frac{\rho}{\epsilon_0} \quad \text{with } \rho \text{ the volume density of charges.} \tag{3}$$

To gain inspiration, we consider the following examples:

- The electric field between the plates of a capacitor is constant and orthogonal to the plates, so the corresponding potential is $\varphi(\vec{r}) = A\ x = A\ r\cos(\theta)$.

- For a pointcharge (a monopole), $\varphi(\vec{r}) = \frac{B}{r}$.

- For a dipole, $\varphi(\vec{r}) = \frac{C\cos(\theta)}{r^2}$.

In these three cases, the charges do not have a volume density (the capacitor has a surface density) so that one has actually solved the Laplace equation.

(a) (1p) Write the boundary condition at infinity $\vec{u} = U\ \vec{e}_z$ in polar coordinates *i.e.* find coefficients $\alpha(\theta)$ and $\beta(\theta)$ such that:

$$\lim_{r\to+\infty} \vec{u}(r, \theta) = U\ \Big(\alpha(\theta)\ \vec{e}_r + \beta(\theta)\ \vec{e}_\theta\Big) \tag{4}$$

(b) (1p) One looks for solutions of equation (1) satisfying the boundary conditions (2) and (4). Why is the following expansion a solution of the equation (1)?

$$\phi(r, \theta) = A\ r\cos(\theta) + \frac{B}{r} + \frac{C\cos(\theta)}{r^2} \tag{5}$$

Find the coefficients A, B and C fitting the boundary conditions.

(c) (1p+1bp) Give the expression of the velocity field and plot it. What is happening for $\theta = 0$ and $\theta = \pi$? *Bonus question*: Does this velocity field make sense physically? Explain why.

2 Diffusion equation

You don't need any knowledge in fluid dynamics or elasticity theory to solve this exercise.

Let us consider a thin elastic film of constant thickness h floating on a viscous fluid layer, which is itself supported on a prestretched rubber sheet. Initially, the thin film of length $2L$ is flat and the viscous layer has height H_0. We are interested in the buckling instability of the thin film when the prestretch in the rubber is released (see figure 1).

(a) (1p) The thickness of the viscous layer is supposed to be very small compared to the length of the elastic film *viz.* $H_0 \ll L$. This is the lubrication approximation, under which the Navier-Stokes equation reduces to:

$$\frac{\partial P}{\partial x} = \eta \, \frac{\partial^2 v_x}{\partial z^2} \,, \tag{6}$$

where P is the pressure, η the constant dynamic viscosity and v_x the x-component of velocity. Assuming a constant pressure gradient $\frac{\partial P}{\partial x}$, solve this PDE for $v_x(x,z,t)$ with the following boundary conditions:
- Bottom: no-slip at the interface fluid/rubber $z = 0$, so $v_x(x,0,t) = \dot{\epsilon}x,\ \forall (x,t) \in [-L,L] \times \mathbb{R}^+$ where $\dot{\epsilon}$ is the strain rate of the rubber contraction.
- Top: at the interface film/fluid $z = H_0$ a shear stress $T(x,t)$ is transferred to the elastic film, whence $\left.\frac{\partial v_x}{\partial z}\right|_{z=H_0} = \frac{T(x,t)}{\eta},\ \forall (x,t) \in [-L,L] \times \mathbb{R}^+$.

Express your solution in terms of $T(x,t)$, $\frac{\partial P}{\partial x}$ and the other constants that were introduced.

(b) (1p) The displacement field $u(x,t)$ along x-axis of the interface film/fluid is related to the velocity field that you have just obtained in the following way:

$$\frac{\partial u}{\partial t} = v_x(x, H_0, t) \tag{7}$$

Derive a differential equation for $u(x,t)$ under the assumption that the pressure gradient is actually zero. Use the relation $T(x,t) = \alpha \, \eta \, \frac{\partial^2 u}{\partial x^2}$ from linear elasticity. You should get a diffusion equation with a source term.

(c) (3p) Solve the PDE you found in (b), i.e. $\frac{\partial u}{\partial t} = \alpha H_0 \frac{\partial^2 u}{\partial x^2} + \dot{\epsilon}x$, by separating variables with the initial condition $u(x,0) = 0,\ \forall x \in [-L,L]$ and the boundary condition $\left.\frac{\partial u}{\partial x}\right|_{x=\pm L} = 0, \forall t \geq 0$. You will need to look for a series solution in order to satisfy the initial condition. Detail the calculation of the series' coefficients.

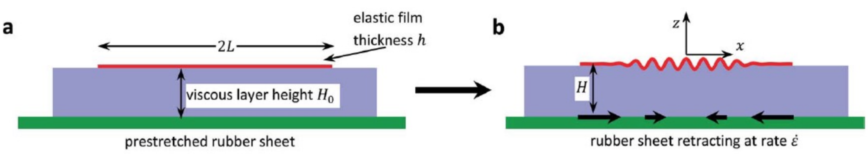

Figure 1: Buckling instability of a thin film

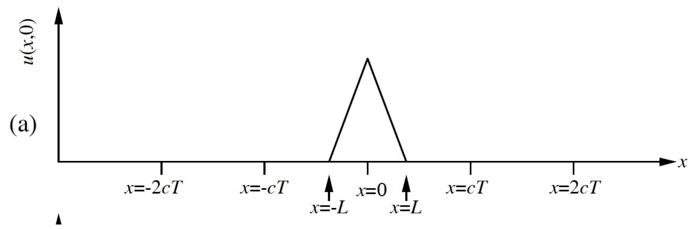

Figure 2: Triangle wave at $t = 0$

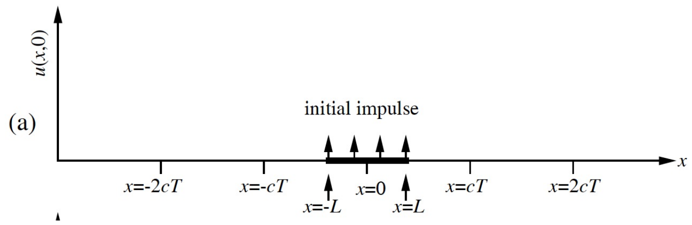

Figure 3: Initial impulse

3 Wave equation

The linear 1D wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c = cst \tag{8}$$

with initial conditions of the form

$$\begin{cases} u(x,0) = q(x) \\ \frac{\partial u}{\partial t}(x,0) = p(x) \end{cases} \qquad \forall x \in \mathbb{R} \ . \tag{9}$$

The solution of this initial value problem is given by the so-called D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\Big(q(x+ct) + q(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(x')dx' \tag{10}$$

(a) (1p) Apply this formula to the triangular wave, defined as follows (see figure 2):

$$\begin{cases} q(x) = \left(1 - \frac{|x|}{L}\right)\ \theta\left(1 - \frac{|x|}{L}\right) \\ p(x) = 0 \end{cases} \qquad \forall x \in \mathbb{R} \tag{11}$$

where θ is the Heaviside step function. Plot your solution as a function of x at time t such that $ct > L$; explain the correspondence between the analytic expression and the graph.

(b) (2p) The same question as (a), but now for the initial impulse (see figure 3):

$$\begin{cases} q(x) = 0 \\ p(x) = \theta(x+L)\theta(L-x) \end{cases} \qquad \forall x \in \mathbb{R} \tag{12}$$

Plot your solution as a function of x at time t such that $ct > L$; explain the correspondence between the analytic expression and the graph.

Hint: A primitive of p is $g(x) = (x+L)\ \theta(x+L)\theta(L-x) + 2L\ \theta(x-L)$; verify this result.

(c) (1bp) *Bonus question*: What is the difference between the propagation of a triangular wave and that of an initial impulse? Think about the concept of the domain of dependence in D'Alembert's formula.