Mathematical Methods for Physicists, FK7048, fall 2019
Exercise sheet 2, **due Monday september 16th, 10:00**

Total amount of points: 13 p, and 1 bonus points (1 bp).

1 Reduction of order

(2p) For a linear ODE of any order, if you know one solution y_1 then you can reduce the order of the ODE by performing the substitution $y = y_1 u$. Do this for a general 2nd order ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

Solve the equation that you get for u. You should eventually find back the integral expression seen in class of the second solution generated from the first.

2 A scale invariant equation

An ODE is said to be scale invariant if there exists a $p \in \mathbb{R}$ such that the transformation $(x \to ax; y \to a^p y)$ leaves the ODE unchanged. This is a generalization of isobaric equations (which are 1st order ODEs). For a scale invariant ODE, one can perform the transformation $y(x) = x^p u(x)$ to obtain an ODE for $u(x)$ which is, by construction, equidimensional in x. That is, an equation that is invariant under the transformation $x \to ax$. It is known that an equidimensional equation can be transformed into an autonomous equation for $u(t)$ thanks to the change of variable $x = e^t$. We remind that, by definition, an autonomous ODE does not depend explicitly on the independent variable (here t). This means that it is possible to reduce its order by setting $w(u) = u'(t)$. This yields a simpler ODE for $w(u)$, that does not explicitly depend on t.

(a) (4p) Apply this recipe the following scale invariant equation:

$$xy''(x) + \big(2 - xy(x)\big)y'(x) = y^2(x) \tag{2}$$

to find the differential equation for $w(u)$, namely $ww' - w = uw$.

(b) (1bp) Perform the integrals, to show that the general solution $y(x)$ can be written as

$$y(x) = \frac{c_1}{x}\tan\big[(\frac{c_1}{2}\ln x + c_2)\big] - \frac{1}{x} \ .$$

3 Equivalence between 2nd order linear ODEs and Riccati equations

Any 2nd order linear ODE is equivalent to a Riccati equation. We are going to use this fact to solve the following equation:

$$y''(x) + 2xy'(x) + (2x+1)y(x) = 0 \ . \tag{3}$$

(a) (1 p) Perform the transformation $y(x) = e^{u(x)}$ and show that $v(x) = u'(x)$ satisfies a Riccati equation.

(b) (1 p) Look for a particular solution of this Riccati equation which has the simple form $v_0(x) = ax + b, \quad (a, b) \in \mathbb{C}^2$.

(c) (1 p) Substitute $v(x) = w(x) + v_0(x)$ in the Riccati equation. You should recognize the kind of equation that you get for $w(x)$. Apply the trick seen in class to transform this last equation into a 1st order linear non homogeneous ODE.

(d) (1 p) Solve this non homogeneous ODE by the method of the integrating factor. Write your solution in terms of the function $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt$. One can now write the solution for (3) as an integral expression, but you do not have to do this.

4 Nuclear disintegration

(3p) Consider N_0 unstable nuclei of type A that disintegrates into nuclei of type B with a rate λ_A. Nuclei of type B are themselves unstable and disintergrate into nuclei of type C with a rate λ_B. The dynamics of the reaction is governed by the following differential equations:

$$\begin{cases} \frac{dN_A}{dt} = -\lambda_A N_A \\ \frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A \end{cases} \tag{4}$$

Find the number $N_B(t)$ of nuclei of type B at time t using an integrating factor.