# Tutorial Class 9
# Mathematical Methods in Physics

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1. **P3 2005-05-30.** The wave function for the harmonic oscillator in its ground state is given by

$$\psi(x) = \frac{1}{\sqrt{a\sqrt{\pi}}} \exp\left(-\frac{x^2}{2a^2}\right). \tag{1}$$

Use the Fourier transform to convert this wave function onto a wave function in $k$-space, i.e. find $\tilde{\psi}(k)$.
*Hint:* Choose a suitable path in the complex plane for the integration.

**Solution.** The Fourier transform of the position space wave function is given by

$$\begin{aligned}\tilde{\psi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{ikx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2}+ikx}\, dx\end{aligned} \tag{2}$$

We complete the square in the exponent and find

$$\begin{aligned}-\frac{x^2}{2a^2} + ikx &= -\frac{1}{2a^2}(x^2 - 2ika^2 x) \\ &= -\frac{1}{2a^2}\left(x - ika^2\right) - \left(-\frac{1}{2a^2}\right)\left(-ika^2\right)^2 \\ &= -\frac{1}{2a^2}\left(x - ika^2\right) - \frac{k^2a^2}{2}\end{aligned} \tag{3}$$

so the Fourier transformed wave function is

$$\tilde{\psi}(k) = \frac{e^{-\frac{k^2a^2}{2}}}{\sqrt{2\pi}\sqrt{a\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-\frac{(x-ika^2)^2}{2a^2}}\, dx. \tag{4}$$

We now make the variable change

$$\begin{aligned}&\frac{(x - ika^2)}{\sqrt{2a^2}} = t, \quad \frac{dx}{\sqrt{2a^2}} = dt \\ &x \to \pm\infty \Rightarrow t \to \pm\infty - i\epsilon, \quad \epsilon = \frac{ka}{\sqrt{2}}\end{aligned} \tag{5}$$

which transforms the integral into

$$\tilde{\psi}(k) = \frac{\sqrt{2a^2}e^{-\frac{k^2a^2}{2}}}{\sqrt{2\pi a\sqrt{\pi}}} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{-t^2}\, dt \tag{6}$$

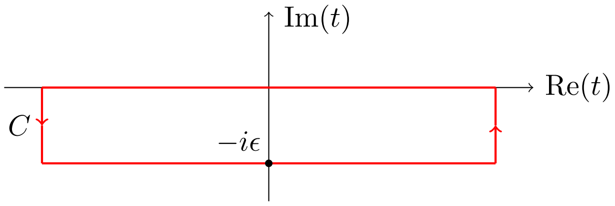


Figure 1: Contour in P3 2005-05-30. Note that $-i\epsilon$ is not a pole.

We do the integral using contour integration techniques. We integrate over a positively oriented, closed contour $C$ that starts by going from $t = -\infty - i\epsilon$ to $t = +\infty - i\epsilon$, then goes up to the real axis at $t = +\infty$, along the real axis to $t = -\infty$ and back down to the point $t = -\infty - i\epsilon$ (see Fig. 1). The contour does not enclose any pole of the integrand (in fact the integrand is an entire function of $t$ and has no poles in the complex plane), therefore the total integral over $C$ is zero. The integrand goes to zero for $|t| \to \infty$, so the integrations along the two pieces that run parallell to the imaginary axis both become zero. We then have (writing $t = u + iv$)

$$\oint_C e^{-t^2}\, dt = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{-t^2}\, dt + \int_{\infty}^{-\infty} e^{-u^2}\, du = 0 \tag{7}$$

so that

$$\int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{-t^2}\, dt = \int_{-\infty}^{\infty} e^{-u^2}\, du. \tag{8}$$

The righthand integral is a standard Gaussian integral with value

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi} \tag{9}$$

and therefore the Fourier transformed wave function is

$$\begin{aligned}\tilde{\psi}(k) &= \frac{\sqrt{2a^2}e^{-\frac{k^2a^2}{2}}}{\sqrt{2\pi a\sqrt{\pi}}}\sqrt{\pi} \\ &= \frac{\sqrt{a}}{\pi^{1/4}}e^{-\frac{k^2a^2}{2}} \end{aligned} \tag{10}$$

which is again a Gaussian, but of width inversely related to $a$ compared to $\psi(x)$ (large $a$ means that $\psi(x)$ is narrow while $\tilde{\psi}(k)$ is wide and vice versa).

**Note:** The Gaussian integral $I$ can be done in several ways. One way is to write

$$I \equiv \int_{-\infty}^{\infty} e^{-t^2}\, dt = 2\int_{0}^{\infty} e^{-t^2}\, dt = 2\int_{0}^{\infty} e^{-z}\, \frac{dz}{2z^{1/2}} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{11}$$

where we have made the variable transform $z = t^2$ and recognised the integral as a Gamma function. We can also solve the Gaussian integral by letting $t = r$ where $r$ is the radial distance $r$ in polar coordinates. Since $r^2 = x^2 + y^2$ and $dx\,dy = r\sin\theta\, d\theta\, dr = 2\pi r\, dr$ since the integrand is independent of $\theta$ we have

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = 2\pi \int_{0}^{\infty} e^{-r^2} r\, dr = 2\pi \int_{0}^{\infty} \frac{e^{-s}}{2}\, ds = \pi \tag{12}$$

and therefore $I = \sqrt{\pi}$

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2. **P10 2013-11-09.** The nuclear form factor $F(\vec{k})$ and the charge distribution $\rho(\vec{r})$ are 3D Fourier transforms of each other

$$F(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int \rho(\vec{r}) e^{i\vec{k}\cdot\vec{r}}\, d^3r. \tag{13}$$

Show that, if the measured form factor is obtained as $F(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\left(1 + \frac{k^2}{a^2}\right)^{-1}$ then the underlying charge distribution is $\rho(\vec{r}) = \frac{a^2}{4\pi}\frac{e^{-ar}}{r}$

**Solution.** Since $F(\vec{k})$ is the Fourier transform of $\rho(\vec{r})$, $\rho(\vec{r})$ is the *inverse* transform of $F(\vec{k})$,

$$\rho(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int F(\vec{k}) e^{-i\vec{k}\cdot\vec{r}}\, d^3k \tag{14}$$

where the integral is over all of $k$-space. We write the integral in spherical coordinates so that $d^3k = k^2 dk d(\cos\theta) d\varphi$ and $\vec{k}\cdot\vec{r} = kr\cos\theta$ where we have aligned our coordinate system so that $\vec{k}$ is parallell to the $z$ axis and the angle between $\vec{k}$ and $\vec{r}$ hence coincides with the polar angle $\theta$. There is no dependence on the azimuthal angle $\varphi$, that part of the integration therefore just gives a factor $2\pi$ and we are left with

$$\begin{aligned}
\rho(\vec{r}) &= \frac{2\pi}{(2\pi)^{3/2}} \int_{k=0}^{k=\infty} \int_{\cos\theta=-1}^{\cos\theta=1} \frac{e^{-ikr\cos\theta}}{1+\frac{k^2}{a^2}} k^2 \; dk \, d(\cos\theta) \\
&= \frac{a^2}{4\pi^2} \int_0^\infty \frac{k^2}{a^2+k^2} \left[ -\frac{e^{-ikr\cos\theta}}{ikr} \right]_{\cos\theta=-1}^{\cos\theta=1} dk \\
&= \frac{a^2}{4\pi^2 ri} \int_0^\infty \frac{k}{a^2+k^2} \left( e^{ikr} - e^{-ikr} \right) \; dk \\
&= \frac{a^2}{4\pi^2 ri} \left( \int_0^\infty \frac{ke^{ikr}}{a^2+k^2} \; dk - \int_0^{-\infty} \frac{(-k)e^{+ikr}}{a^2+(-k)^2} \; (-dk) \right) \\
&= \frac{a^2}{4\pi^2 ri} \int_{-\infty}^\infty \frac{ke^{ikr}}{a^2+k^2} \; dk. \qquad (15)
\end{aligned}$$

We solve the integral using contour integration, with a contour $C$ that is the integration along the real axis and a semi-circle in the upper half-plane. Since $k > 0$ and the integrand without the exponential vanishes for large $k$ we can use Jordan's lemma to show that the integration along the circular arc gives zero contribution to the integral. The integrand has poles at $k = \pm ia$ and the contour $C$ encloses the pole at $k = +ia$, so we get from the residue theorem

$$\begin{aligned}
\int_{-\infty}^\infty \frac{ke^{ikr}}{a^2+k^2} \; dk &= 2\pi i \,\mathrm{Res}\left( \frac{ke^{ikr}}{a^2+k^2}, k = ia \right) \\
&= 2\pi i \lim_{k\to ia} (k - ia) \frac{ke^{ikr}}{(k-ia)(k+ia)} \\
&= 2\pi i \, \frac{iae^{i^2 ar}}{2ia} \\
&= \pi i e^{-ar} \qquad (16)
\end{aligned}$$

and the charge density therefore becomes

$$\begin{aligned}
\rho(\vec{r}) &= \frac{a^2}{4\pi^2 ri} \, \pi i e^{-ar} \\
&= \frac{a^2}{4\pi} \frac{e^{-ar}}{r} \qquad (17)
\end{aligned}$$

which was what we wanted to show.

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3. **P7 2013-01-03.** An oscillator is subject to both a dissipation and a driving force $f(t)$ where $f(t) = \gamma \exp(-t)$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. The equation describing the subsequent motion can be written

$$\frac{d^2}{dt^2} X(t) + 2\beta \frac{d}{dt} X(t) + \omega_0^2 X(t) = f(t). \qquad (18)$$

Use the Fourier transform $\tilde{g}(\omega) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} G(t) \exp(i\omega t)\, dt$ to show that the retarded Green's function $G_r(t)$ is given by

$$G_r(t,t') = \begin{cases} 0, & t < t' \\ (1/\omega_1) \exp(-\beta(t-t')) \sin(\omega_1(t-t')), & t > t' \end{cases} \tag{19}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. (Retarded Green's function has the cause preceding the effect.)

**Solution.** The Green's function satisfies the differential equation

$$\frac{d^2}{dt^2}G(t,t') + 2\beta\frac{d}{dt}G(t,t') + \omega_0^2 G(t,t') = \delta(t-t'). \tag{20}$$

The Fourier transform of $G(t,t')$ with respect to $t$ is

$$\tilde{g}(\omega,t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t,t')e^{i\omega t}\, dt \tag{21}$$

and derivatives of $G(t,t')$ become simple in Fourier space. We have, using the inverse transform formula

$$\begin{aligned} \frac{d}{dt}G(t,t') &= \frac{d}{dt}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\omega,t')e^{-i\omega t}\, d\omega\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\omega,t')\frac{d}{dt}e^{-i\omega t}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-i\omega)\tilde{g}(\omega,t')e^{-i\omega t}\, d\omega \end{aligned} \tag{22}$$

so the inverse transform of $(-i\omega)\tilde{g}$ is $dG/dt$ and hence the Fourier transform of $dG/dt$ is $(-i\omega)\tilde{g}$. In the same way the Fourier transform of $d^2G/dt^2$ is $(-i\omega)^2\tilde{g} = -\omega^2\tilde{g}$. The Fourier transform of the left-hand side of the Green's function differential equation is therefore

$$\mathcal{L}\left[\text{LHS}\right] = \left(-\omega^2 - 2i\beta\omega + \omega_0^2\right)\tilde{g}(\omega,t') \tag{23}$$

The Fourier transform of $\delta(t-t')$ is

$$\begin{aligned} \mathcal{L}\left[\delta(t-t')\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t-t')e^{i\omega t}\, dt \\ &= \frac{e^{i\omega t'}}{\sqrt{2\pi}} \end{aligned} \tag{24}$$

and when we Fourier transform both sides of the Green's function differential equation we therefore get

$$\left(-\omega^2 - 2i\beta\omega + \omega_0^2\right)\tilde{g}(\omega,t') = \frac{e^{-i\omega t}}{\sqrt{2\pi}} \tag{25}$$

where we note that the factor multiplying $\tilde{g}$ on the left-hand side is now simply an algebraic expression. We get

$$\tilde{g}(\omega, t') = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t'}}{\omega_0^2 - \omega^2 - 2i\beta\omega} \tag{26}$$

and inverse transforming to get $G(t, t')$ we find

$$\begin{aligned} G(t, t') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(\omega, t') e^{-i\omega t} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega_0^2 - \omega^2 - 2i\beta\omega} \, d\omega \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2} \, d\omega. \end{aligned} \tag{27}$$

We will solve the integral using contour integration in the complex plane. The integrand has poles at

$$\omega^2 + 2i\beta\omega - \omega_0^2 = 0 \quad \Rightarrow \quad \omega = -i\beta \pm \sqrt{\omega_0^2 - \beta^2} \equiv -i\beta \pm \omega_1 \tag{28}$$

where we have defined the constant $\omega_1$ in the last equality. We note that the sign in the exponent depends on the sign of $t - t'$ and therefore separate into two cases: $t - t' < 0$ and $t - t' > 0$. We note also that the remaining part of integrand without the exponential goes to zero for large $|\omega|$.

$t - t' < 0$ We close the contour in the upper half-plane with a semi-circle and call it $C$. The contour then encloses no poles, since both poles are in the lower half-plane. The factor multiplying $\omega$ in the exponent is now positive and the remaining part of the integrand goes to zero for large $|\omega| \to \infty$ so by Jordan's lemma the integration along the semi-circle is zero and therefore

$$\oint_C \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2} \, d\omega = 0, \quad t - t' < 0 \tag{29}$$

so that $G(t, t') = 0$ for $t - t' < 0$.

$t - t' > 0$ We now close the contour instead in the lower half-plane with a semi-circle, *negatively* oriented (to make the integration along the real axis in the correct direction), and call the contour $C'$. The contour then encloses the two poles at $\omega = -i\beta \pm \omega_1$. The effect of having the contour in the lower halfplane is that $\sin\theta$ is negative (since then $\pi \leq \theta \leq 2\pi$). Together with the opposite sign of $t - t'$ compared to above this results in the same conditions as for the standard application of

Jordan's lemma (i.e. an exponent with a positive constant)[1]. Therefore we have

$$\oint_{C'} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}\, d\omega = -2\pi i \sum_i \mathrm{Res}\left(\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}, \omega_i\right) \tag{30}$$

with the minus sign on the RHS coming from the negative orientation of $C'$. The contribution along the circular part is again zero from the "extended version" of Jordan's lemma. Call this part of the integral $I_R$, then, taking the absolute value, we have (noting that $C'$ is negatively oriented)

$$|I_R| = \int_{2\pi}^{\pi} \frac{e^{R(t-t')\sin\theta}}{|R^2 e^{2i\theta} + 2i\beta R e^{i\theta} - \omega_0^2|} R d\theta \tag{31}$$

and since $t-t' > 0$ and $\sin\theta < 0$ for $\pi < \theta < 2\pi$ the exponent is always negative, so that the integrand goes to zero as $R \to \infty$ (the part without the exponential goes to zero since the denominator is $\mathcal{O}(R^2)$ and the numerator $\mathcal{O}(R)$) and the integration along the semicircle therefore contributes nothing. The conclusion is that the integral along the real axis is the sum of the enclosed residues at $\omega_a = -i\beta + \omega_1$ and $\omega_b = -i\beta - \omega_1$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}\, d\omega &= \oint_{C'} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}\, d\omega \\ &= -2\pi i \sum_{j=a,b} \mathrm{Res}\left(\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}, \omega_j\right).\end{aligned} \tag{32}$$

The residues are

$$\mathrm{Res}_{\omega=-i\beta+\omega_1} = \lim_{\omega\to -i\beta+\omega_1} \frac{e^{-i\omega(t-t')}}{\omega - (-i\beta - \omega_1)} = \frac{e^{-\beta(t-t')} e^{-i\omega_1(t-t')}}{2\omega_1} \tag{33}$$

$$\mathrm{Res}_{\omega=-i\beta-\omega_1} = \lim_{\omega\to -i\beta-\omega_1} \frac{e^{-i\omega(t-t')}}{\omega - (-i\beta + \omega_1)} = \frac{e^{-\beta(t-t')} e^{+i\omega_1(t-t')}}{-2\omega_1} \tag{34}$$

and their sum becomes

$$\begin{aligned}\sum_i \mathrm{Res}\left(\frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}, \omega_i\right) &= -\frac{e^{-\beta(t-t')}}{2\omega_1}\left(e^{i\omega_1(t-t')} - e^{-i\omega(t-t')}\right) \\ &= -\frac{ie^{-\beta(t-t')}}{\omega_1} \sin\left(\omega_1(t-t')\right).\end{aligned} \tag{35}$$

The integral becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t')}}{\omega^2 + 2i\beta\omega - \omega_0^2}\, d\omega &= (-2\pi i)\left(-\frac{ie^{-\beta(t-t')}}{\omega_1} \sin\left(\omega_1(t-t')\right)\right) \\ &= -\frac{2\pi e^{-\beta(t-t')}}{\omega_1} \sin\left(\omega_1(t-t')\right), \quad t - t' > 0\end{aligned} \tag{36}$$

[1] That is, we can extend Jordan's lemma to account also for negative constants in the exponents remembering that in this case we need to take a contour that is a semicircle in the *lower* halfplane.

and the Green's function is therefore (note the relative factor $-1/2\pi$ between the Green's function and the integral, see Eq. (27))

$$G(t,t') = \begin{cases} 0, & t < t' \\ \dfrac{e^{-\beta(t-t')}}{\omega_1} \sin\big(\omega_1(t-t')\big), & t > t' \end{cases} \tag{37}$$

and since the Green's function represents a point source perturbation at $t'$ this indeed has the cause preceding the effect since it is zero for $t < t'$ and is therefore the retarded Green's function.

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4. **P6 2015-01-02.** Use Laplace transforms to solve the set of equations subject to $y(0) = 1$, $z(0) = 0$ with

$$\frac{dy}{dt} - 2y + z = 0$$
$$\frac{dz}{dt} - y - 2z = 0.$$

**Solution.** The Laplace transform of a function $y(t)$ is given by

$$\mathcal{L}[y(t)] = \tilde{y}(s) = \int_0^\infty y(t)e^{-st}\,dt \tag{38}$$

and it can be shown (try it!) that the transform of a derivative is given by

$$\mathcal{L}[y'(t)] = s\tilde{y}(s) - y(0). \tag{39}$$

The Laplace transformed system of ODE:s then becomes

$$\begin{cases} s\tilde{y}(s) - y(0) - 2\tilde{y}(s) + \tilde{z}(s) = 0 \\ s\tilde{z}(s) - z(0) - \tilde{y}(s) - 2\tilde{z}(s) = 0 \end{cases} \tag{40}$$

and inserting the given initial conditions $y(0) = 1$, $z(0) = 0$ this can be written

$$\begin{cases} (s-2)\tilde{y}(s) + \tilde{z}(s) = 1 \\ (s-2)\tilde{z}(s) - \tilde{y}(s) = 0 \end{cases} . \tag{41}$$

The first of these equations gives $\tilde{z} = 1 - (s-2)\tilde{y}$ and inserting this in the second equation we find

$$\begin{aligned} &(s-2)\left(1 - (s-2)\tilde{y}\right) - \tilde{y} = 0 \\ \Rightarrow \quad &\left((s-2)^2 + 1\right)\tilde{y} = s - 2 \\ \Rightarrow \quad &\tilde{y}(s) = \frac{s-2}{(s-2)^2+1} \end{aligned} \tag{42}$$

i.e.

$$\tilde{z}(s) = 1 - \frac{(s-2)^2}{(s-2)^2+1} = \frac{1}{(s-2)^2+1} \tag{43}$$

We now need to inverse transform to find the solutions $z(t)$ and $y(t)$ to our original system of ODE:s. In principle this can be done with the Bromwich integral using residue calculus but in this case we can find the inverse transforms by looking in a table of transforms like table 20.1 on p. 1012 in AWH and using the general properties of Laplace transforms. In table 20.1 in AWH we find that

$$\mathcal{L}[\cos(t)] = \frac{s}{s^2+1}, \quad \mathcal{L}[\sin(t)] = \frac{1}{s^2+1} \tag{44}$$

which is close to the expressions for $\tilde{z}$ and $\tilde{y}$. These are shifted in their argument and evaluated at $s-2$ instead of $s$, so we need to know what the properties of the Laplace transform are under such a shift. Under a substitution $s \to s-a$ so that the Laplace transform is evaluated instead at $s-a$ a Laplace transform $f(s)$ of a function $F(t)$ becomes,

$$f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)\ dt = \int_0^\infty e^{-st} \left(e^{at} F(t)\right)\ dt = \mathcal{L}[e^{at}F(t)] \tag{45}$$

or in other words

$$\mathcal{L}^{-1}[f(s-a)] = e^{at}F(t) \tag{46}$$

Using this property we can in our case identify the expressions for $\tilde{z}(s)$ and $\tilde{y}(s)$ as the transforms of

$$y(t) = \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1}\right] = e^{2t}\cos t \tag{47}$$

$$z(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}\sin t \tag{48}$$

which are therefore the solutions to our system of ODE:s.

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