Tutorial Class 7
Mathematical Methods in Physics

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Last typeset: October 11, 2018

1. **P2 2011-10-29** Express the polynomial $7x^4 - 3x + 1$ as a linear combination of Legendre polynomials.

   Solution. The Legendre polynomials P_ℓ are a *complete* and *linearly independent* set of functions, meaning that any polynomial $R(x)$ of degree k can be expressed as a linear combination of Legendre polynomials with degree $\ell \leq k$. In other words, the $\{P_\ell(x)\}$ are a *basis* which a polynomial can be expanded in. We have

$$R(x) = \sum_{\ell=0}^{k} a_\ell P_\ell(x) \tag{1}$$

   which then corresponds the change of basis from the basis spanned by $\{1, x, x^2, x^3, x^4\}$ to the one spanned by $\{P_\ell(x) : \ell = 0, 1, 2, 3, 4\}$ since $R(x)$ is here a polynomial of degree 4. We have then

$$R(x) = 7x^4 - 3x + 1 = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4 \tag{2}$$

   and using the explicit expressions for the P_ℓ (see e.g. table 15.1 on p. 719 in AWH, they can also be obtained using the Rodrigues formula) given by

$$P_0(x) = 1 \tag{3}$$
$$P_1(x) = x \tag{4}$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{5}$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{6}$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \tag{7}$$

   we have

$$\begin{aligned} 7x^4 - 3x + 1 = &\left(a_0 - \frac{a_2}{2} + \frac{3a_4}{8}\right) + \left(a_1 - \frac{3a_3}{2}\right) x \\ &+ \left(\frac{3a_2}{2} - \frac{30a_4}{8}\right) x^2 + \frac{5a_3}{2} x^3 + \frac{35a_4}{8} x^4. \end{aligned} \tag{8}$$

Comparing coefficients on the LHS and RHS we then find

$$\begin{cases} a_0 - \dfrac{a_2}{2} + \dfrac{3a_4}{8} = 1 \\ a_1 - \dfrac{3a_3}{2} = -3 \\ \dfrac{3a_2}{2} - \dfrac{30a_4}{8} = 0 \\ \dfrac{5a_3}{2} = 0 \\ \dfrac{35a_4}{8} = 7 \end{cases} \tag{9}$$

and solving for the a_ℓ we find

$$\begin{cases} a_0 = \dfrac{12}{5} \\ a_1 = -3 \\ a_2 = 4 \\ a_3 = 0 \\ a_4 = \dfrac{8}{5} \end{cases} \tag{10}$$

so that

$$7x^4 - 3x + 1 = \frac{12}{5}P_0 - 3P_1 + 4P_2 + \frac{8}{5}P_4. \tag{11}$$

Note that we must have a term with P_4 in our expression since this is the first P_ℓ for which there is a term proportional to x^4, and that there is no term with P_3 since there was no term proportional to x^3 in $R(x)$.

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2. **P7 2009-10-24.** Use the series definition of the Legendre functions

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{12}$$

to explicitly verify the recursion relation

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x). \tag{13}$$

Here, $[n/2]$ stands for the largest integer $\leq n/2$.
Hint: For n even, i.e. $n = 2p$, $[(n+1)/2 = [n/2] = p$. For n odd, i.e. $n = 2p+1$,

$[(n+1)/2] = p+1 \neq [n/2] = p$ but the extra terms with $k = p+1$ vanish due to the factorial $(-1)! \to \infty$ in the denominator.

Solution. We begin by writing out the terms in the RHS explicitly:

$$(n+1)P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+1)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{14}$$

$$\begin{aligned} xP_n'(x) &= x \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k-1} \\ &= \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}. \end{aligned} \tag{15}$$

The RHS is then

$$(n+1)P_n(x) + xP_n'(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n+1-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}. \tag{16}$$

We now write out and simplify the LHS:

$$\begin{aligned} P_{n+1}'(x) &= \frac{d}{dx} \sum_{k=0}^{[(n+1)/2]} (-1)^k \frac{(2n+2-2k)!}{2^{n+1} k!(n+1-k)!(n+1-2k)!} x^{n+1-2k} \\ &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \frac{(n+1-2k)(2n+2-2k)!}{2^{n+1} k!(n+1-k)!(n+1-2k)!} x^{n-2k} \\ &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \frac{(2n+2-2k)(2n+1-2k)(2n-2k)!}{2^{n+1} k!(n+1-k)!(n-2k)!} x^{n-2k} \\ &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \frac{(2n+1-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}. \end{aligned} \tag{17}$$

This final expression is very similar to what we want to show, except that the sum goes to $(n+1)/2$ instead of $n/2$ as in Eq. (16). We now use what is stated in the hint.

For n even, i.e. $n = 2p$ for some integer p, we have that the final value of the summation index is

$$[(n+1)/2] = [p+1/2] = p = [n/2] \tag{18}$$

where, to reiterate, we use the notation that $[(n+1)/2]$ is equal to the largest integer $\leq (n+1)/2$. This means that for n even we can replace the final value of the summation index in Eq. (17) with $[n/2]$ and the proof is done.

For n odd, i.e. $n = 2p + 1$ for some integer p we have instead that the final value of the summation index is

$$[(n+1)/2] = [p+1] \neq [n/2] = p \tag{19}$$

To complete our proof we write out explicitly the final term in the sum on the LHS for the case of n being odd. For n odd $(n+1)/2$ is an integer, so $k = (n+1)/2$ will be the final value of the summation index in the sum, we have then inserting $k = (n+1)/2$

$$\begin{aligned}
\text{final term, } n \text{ odd} &= (-1)^{(n+1)/2} \frac{(2n+1-(n+1))(2n-(n+1))!}{2^n \left(\frac{n+1}{2}\right)! \left(n - \frac{n+1}{2}\right)! \left(n-(n+1)\right)!} x^{n-(n+1)} \\
&= (-1)^{(n+1)/2} \frac{n(n-1)!}{2^n \left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)! (-1)!} \frac{1}{x} \\
&\to 0
\end{aligned} \tag{20}$$

where we have used that $(-1)! \to \infty$, which for example can be seen by using the fact that $(-1)! = \Gamma(0)$ and $z = 0$ is a pole of the Γ function.[1] Therefore, the last term in the sum is zero and we can write, for n both even and odd

$$\begin{aligned}
P'_{n+1}(x) &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \frac{(2n+1-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \\
&= \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n+1-2k)(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \\
&= (n+1) P_n(x) + P'_n(x)
\end{aligned} \tag{22}$$

which completes our proof.

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3. **P4 2015-01-02.** A plane wave may be expanded in a series of spherical waves by the Rayleigh equation

$$e^{ikr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta). \tag{23}$$

Show that $a_n = i^n(2n+1)$.

[1] We have from the Weierstrass infinite product definition

$$\frac{1}{\Gamma(0)} = 0 \cdot e^0 \prod_{n=0}^{\infty} \left(1 + \frac{0}{n}\right) e^0 = 0 \cdot \prod_{n=0}^{\infty} 1 = 0, \tag{21}$$

so that $\Gamma(0) = (-1)! \to \infty$.

Solution. The j_n are spherical Bessel functions, defined by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) \tag{24}$$

where $J_{n+\frac{1}{2}}$ is a Bessel function of the first kind (of half-integral order, since n is an integer). From Eq. 14.155 in AWH a series expansion of $j_n(x)$ is given by

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s+n+\frac{1}{2}} \\ &= \frac{\sqrt{\pi}x^n}{2^{n+1}} \sum_{s=0} \frac{(-1)^s}{s!\Gamma(s+n+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s} \end{aligned} \tag{25}$$

To find the a_n we begin by using the orthogonality of the Legendre functions, namely that

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \; d\theta = \frac{2}{2n+1}\delta_{mn} \tag{26}$$

Therefore we multiply our original expression in the problem by $P_m(\cos\theta)\sin\theta$ with $m \neq n$ and integrate from 0 to π to get

$$\begin{aligned} \int_0^{\pi} P_m(\cos\theta) e^{ikr\cos\theta} \sin\theta \; d\theta &= \sum_{n=0}^{\infty} a_n j_n(kr) \int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \; d\theta \\ &= \frac{2}{2m+1} a_m j_m(kr) \end{aligned} \tag{27}$$

To get rid of the r-dependence in the $j_m(kr)$ on the right-hand side we differentiate both sides of Eq. (27) m times with respect to $x = kr$ with $x \to 0$. For small x, we can use the limiting expression in Eq. 14.177 in AWH which states that for $x \ll 1$, $j_n(x)$ is approximately given by[2]

$$j_n(x) \approx \frac{x^n}{(2n+1)!!}, \quad x \ll 1. \tag{28}$$

We can take the m derivatives of the spherical Bessel function and let $x \to 0$ to get,

$$\lim_{x\to 0} \frac{d^m}{dx^m} j_m(x) = \frac{m!}{(2m+1)!!} \tag{29}$$

[2]Looking at the series expansion of $j_m(x)$, we see that $j_m(x)$ goes as x^m times the terms in the sum. Each successive term in the sum contributes with a higher power of x and for small x the terms will therefore be smaller and smaller for higher s (note that $x \geq 0$ here since $x = kr$). The limiting expression for $x \ll 1$ is obtained by including only the $s = 0$ term in the sum.

Eq. (27) then becomes after the differentiation and limit-taking

$$\lim_{x\to 0}\frac{d^m}{dx^m}\int_0^\pi P_m(\cos\theta)e^{ix\cos\theta}\sin\theta\ d\theta = \frac{2a_m}{(2m+1)}\frac{m!}{(2m+1)!!} \tag{30}$$

where we ultimately want to find the expression for what a_m is, so we have to calculate what the LHS is.

The same differentiation and limit $x = kr \to 0$ on the LHS of this leads to (note that the exponential term just gives a factor one when the limit is taken)

$$\begin{aligned}
\lim_{x\to 0}\frac{d^m}{dx^m}\int_0^\pi P_m(\cos\theta)e^{ix\cos\theta}\sin\theta\ d\theta &= \lim_{x\to 0}\int_0^\pi i^m\cos^m\theta P_m(\cos\theta)e^{ix\cos\theta}\sin\theta\ d\theta \\
&= i^m\int_0^\pi \cos^m\theta P_m(\cos\theta)\sin\theta\ d\theta \\
&= i^m\int_{-1}^1 t^m P_m(t)\ dt
\end{aligned} \tag{31}$$

where in the last line we have made the variable substitution $t = \cos\theta$ in the integral. To perform this integral is the task of problem 15.1.15 in AWH. It can be done using the Rodrigues formula for the Legendre polynomials and integrating by parts multiple times. The result is

$$\int_{-1}^1 t^m P_m(t)\ dt = \frac{2\,m!}{(2m+1)!!}. \tag{32}$$

Putting this into Eq. (31), we find in the end that Eq. (30) becomes

$$i^m\frac{2\,m!}{(2m+1)!!} = \frac{2a_m}{(2m+1)}\frac{m!}{(2m+1)!!} \tag{33}$$

leading to

$$a_m = i^m(2m+1) \tag{34}$$

which was what we wanted to show.

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4. **P2 2016-01-20.** Write the Legendre polynomial $P_\ell(x)$ given through Rodrigues' formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{35}$$

and use Leibniz' rule to do the explicit differentiation to show that $P_\ell(1) = 1$.

Solution. Leibniz' rule is for differentiating a product is stated in Eq. 15.75 on p. 742 as well as in Eq. 1.81 on p. 36 in AWH, it is given by

$$\frac{d^m}{dx^m}\Big[A(x)B(x)\Big] = \sum_{s=0}^{m} \binom{m}{s} \frac{d^{m-s}A(x)}{dx^{m-s}} \frac{d^s B(x)}{dx^s}. \tag{36}$$

Here we let $(x^2-1)^\ell = (x-1)^\ell (x+1)^\ell$ and apply Leibniz' formula to get

$$\frac{d^\ell}{dx^\ell}\Big[(x-1)^\ell (x+1)^\ell\Big] = \sum_{s=0}^{\ell} \binom{\ell}{s} \frac{d^{\ell-s}}{dx^{\ell-s}}\big[(x-1)^\ell\big] \frac{d^s}{dx^s}\big[(x+1)^\ell\big] \tag{37}$$

We want the value at $x=1$. This means that we are only interested in terms in the sum where there is no factor $(x-1)$ left. Therefore, only the term in the sum where we differentiate $(x-1)^\ell$ exactly ℓ times will contribute to the value of $P_\ell(x)$ at $x=1$, this is the term with $s=0$, all other terms in the sum will contain at least one factor $(x-1)$ and thus give zero at $x=1$. With only the $s=0$ term left we find

$$\begin{aligned}
P_\ell(1) &= \left(\frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right)\Big|_{x=0} \\
&= \left(\frac{1}{2^\ell \ell!} \binom{\ell}{0} \frac{d^\ell}{dx^\ell}\big[(x-1)^\ell\big](x+1)^\ell\right)\Big|_{x=0} \\
&= \left(\frac{1}{2^\ell \ell!} \ell!(x+1)^\ell\right)\Big|_{x=1} \\
&= \frac{1}{2^\ell}(1+1)^\ell \\
&= 1
\end{aligned} \tag{38}$$

where we used that

$$\frac{d^\ell}{dx^\ell}(x-1)^\ell = \ell \frac{d^{\ell-1}}{dx^{\ell-1}}(x-1)^{\ell-1} = \ldots = \ell(\ell-1)\cdots 3\cdot 2 \frac{d}{dx}(x-1) = \ell! \tag{39}$$

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5. **P6 2013-11-09.** A conducting sphere of radius a is divided into two electrically separate hemispheres by a thin insulating barrier at its equator. The top hemisphere is maintained at a potential V_0 and the bottom hemisphere at $-V_0$. Show that the potential *exterior* to the two hemispheres is

$$V(r,\theta) = V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!} \left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta) \qquad (40)$$

where $P_{2s+1}(\cos\theta)$ is the Legendre polynomial of order $2s+1$.

Solution. From electrodynamics we know that a conductor is an equipotential (so that it makes sense that the two separated hemispheres are each kept at a constant potential). Furthermore, the potential V satisfies the Laplace equation in regions without any charge. In this case we have no φ-dependence in the problem (azimuthal symmetry). Therefore, $V = V(r,\theta)$ and exterior to the sphere the potential satisfies $\nabla^2 V(r,\theta) = 0$.

Using the conventional method of separation of variables in spherical coordinates one can show that the solution to Laplace equation with azimuthal symmetry is given by a Legendre series (Eq. 15.42 in AWH)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell(\cos\theta). \qquad (41)$$

(Note that when there is a φ-dependence in the problem, the solution will instead contain *associated* Legendre functions $P_\ell^m(\cos\theta)$, see Eq. 15.41 in AWH.) In this case we are interested in the potential in a region from $r = a$ out to infinite r. The potential becomes infinite at $r \to \infty$ unless we set all A_ℓ to zero (similarly we would instead have had to set all $B_\ell = 0$ if we were interested in a region containing $r = 0$). We are left with

$$V(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) \qquad (42)$$

and we will now apply the boundary condition at $r = a$ to determine the coefficients B_ℓ. We have there

$$V(a,\theta) = \begin{cases} +V_0, & 0 \le \theta < \pi/2 \\ -V_0, & \pi/2 < \theta \le \pi \end{cases}. \qquad (43)$$

We can project out the B_ℓ using the fact that the Legendre polynomials $P_\ell(\cos\theta)$ are orthogonal on the interval $0 \le \theta \le \pi$ with weight $\sin\theta$[3],

$$\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{2\ell+1} \delta_{\ell\ell'}. \qquad (44)$$

[3]The factor $\sin\theta$ is an artefact of the change of variables from x to $\cos\theta$. The $P_\ell(x)$ are orthogonal on the interval $-1 < x < 1$ with unit weight.

Using this we find that

$$
\begin{aligned}
\int_0^\pi V(a,\theta) P_\ell(\cos\theta)\sin\theta \; d\theta &= \sum_{\ell'} B_{\ell'} a^{-\ell'-1} \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta \; d\theta \\
&= \sum_{\ell'} B_{\ell'} a^{-\ell'-1} \frac{2}{2\ell'+1}\delta_{\ell\ell'} \\
&= B_\ell a^{-\ell-1}\frac{2}{2\ell+1} \qquad (45)
\end{aligned}
$$

or

$$
B_\ell = \frac{a^{\ell+1}(2\ell+1)}{2} V_0 \left(\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta \; d\theta - \int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta \; d\theta \right). \qquad (46)
$$

Denoting by I_ℓ the expression in parentheses containing the two integrals we have (performing a change of variables $x = \cos\theta$)

$$
I_\ell = \int_0^1 P_\ell(x) \; dx - \int_{-1}^0 P_\ell(x) \; dx \qquad (47)
$$

$$
(48)
$$

Now let $x \to -x$ (i.e. make a change of variables) in the second integral, then $dx \to -dx$ and we have

$$
\begin{aligned}
I_\ell &= \int_0^1 P_\ell(x) \; dx + \int_{+1}^0 P_\ell(-x) \; dx \\
&= \int_0^1 \left[P_\ell(x) - P_\ell(-x)\right] \; dx \\
&= \begin{cases} 0, & \ell \text{ even} \\ 2\displaystyle\int_0^1 P_\ell(x) \; dx, & \ell \text{ odd} \end{cases} \qquad (49)
\end{aligned}
$$

where we have used the parity of the Legendre polynomials in the last equality ($P_\ell(x) = P_\ell(-x)$ for even ℓ and $P_\ell(x) = -P_\ell(-x)$ for ℓ odd). The integral can be calculated using the recursion relation for the Legendre polynomials given by

$$
(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x) \qquad (50)
$$

together with the values of the $P_{\ell\pm1}$ at $x = 0$ and $x = 1$. We have

$$
P_\ell(1) = 1, \text{ all } \ell, \quad P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}. \qquad (51)
$$

Note that since ℓ is odd, $\ell \pm 1$ is even and the value at $x = 0$ for even order $2n$ is therefore the one we need. This gives, writing $\ell = 2k+1$ (all I_ℓ and hence B_ℓ are zero for even ℓ)

$$
\begin{aligned}
I_{2k+1} = 2\int_0^1 P_{2k+1}(x)\ dx &= \frac{2}{2(2k+1)+1}\left[P_{2k+2}(x)\big|_0^1 - P_{2k}(x)\big|_0^1\right] \\
&= \frac{2}{4k+3}\left[-P_{2k+2}(0) + P_{2k}(0)\right] \\
&= \frac{2}{4k+3}\left[-(-1)^{k+1}\frac{(2k+1)!!}{2k+2)!!} + (-1)^k\frac{(2k-1)!!}{(2k)!!}\right] \\
&= \frac{2}{4k+3}(-1)^k\frac{(2k-1)!!}{(2k)!!}\left[+\frac{2k+1}{2k+2} + 1\right] \\
&= \frac{2}{4k+3}(-1)^k\frac{(2k-1)!!}{(2k)!!}\left[\frac{4k+3}{2k+2}\right] \\
&= 2(-1)^k\frac{(2k-1)!!}{(2k+2)!!}. \qquad (52)
\end{aligned}
$$

We could also use the recursion relation already in Eq. (47) (using then also the value $P_{\ell\pm1}(-1) = (-1)^{\ell\pm1} = 1$ for $\ell \pm 1$ even) to get the same result. So the $B_\ell = B_{2k+1}$ are given by

$$
\begin{aligned}
B_{2k+1} &= \frac{a^{2k+2}(4k+3)}{2}V_0\, 2(-1)^k\frac{(2k-1)!!}{(2k+2)!!} \\
&= a^{2k+2}(4k+3)V_0\,(-1)^k\frac{(2k-1)!!}{(2k+2)!!} \qquad (53)
\end{aligned}
$$

(with all $B_\ell = 0$ for even ℓ) and the potential exterior to the sphere becomes

$$
V(r,\theta) = V_0\sum_{k=0}^{\infty}(4k+3)(-1)^k\frac{(2k-1)!!}{(2k+2)!!}\left(\frac{a}{r}\right)^{2k+2} P_{2k+1}(\cos\theta). \qquad (54)
$$