Tutorial Class 6
Mathematical Methods in Physics

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1. **P7 2015-01-02.** Show that, for integer n,

   (a) $\displaystyle\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$

   (b) $\displaystyle\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{\Gamma(n+\frac{1}{2})}{2a^{n+\frac{1}{2}}} = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

   Solution. We note first of all that the LHS expressions are quite similar to the Euler definition of the Gamma function as a definite integral:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \quad \mathrm{Re}(z) > 0. \tag{1}$$

   We will also need the two following important properties of the Gamma function:

$$\Gamma(n+1) = n! \quad \Leftrightarrow \quad \Gamma(n) = (n-1)! \tag{2}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{3}$$

   (a) We start by making a change of integration variable so that $t = ax^2$, $x = (t/a)^{1/2}$ and $x\,dx = dt/(2a)$. This results in

$$\begin{aligned}\int_0^\infty x^{2n+1}e^{-ax^2}\,dx &= \int_0^\infty \left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right)^{2n} e^{-t}\,\frac{dt}{2a}\\ &= \frac{1}{2a}\frac{1}{a^n}\int_0^\infty t^n e^{-t}\,dt\\ &= \frac{\Gamma(n+1)}{2a^{n+1}}\\ &= \frac{n!}{2a^{n+1}}\end{aligned} \tag{4}$$

   which is what we wanted to show.

(b) We make the same change of variables as above, resulting in

$$
\begin{aligned}
\int_0^\infty x^{2n} e^{-ax^2}\, dx = &= \frac{1}{2a}\int_0^\infty \left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right)^{(2n-1)} e^{-t}\, dt \\
&= \frac{1}{2a^{n+\frac{1}{2}}}\int_0^\infty t^{n-\frac{1}{2}} e^{-t}\, dt \\
&= \frac{1}{2a^{n+\frac{1}{2}}}\Gamma\left(n+\frac{1}{2}\right) \\
&= \frac{1}{2a^{n+\frac{1}{2}}}\frac{\sqrt{\pi}(2n-1)!!}{2^n} \\
&= \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} \qquad (5)
\end{aligned}
$$

where in the next to last equality we have used Eqs. (2) and (3) to obtain

$$
\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) \\
&\vdots \\
&= \overbrace{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\frac{1}{2}}^{n \text{ factors}}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2\cdot 2\cdots 2}\sqrt{\pi} \\
&= \frac{(2n-1)!!}{2^n}\sqrt{\pi} \qquad (6)
\end{aligned}
$$

For completeness we here quote the definitions of the odd and even double factorials:

$$
(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-3)(2n-1) \qquad (7)
$$

$$
(2n)!! = 2\cdot 4\cdot 6\cdots(2n-2)(2n) \qquad (8)
$$

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2. **P6 2011-01-05.** Make the change of variables $z = e^{x^2/2}$ in the differential equation $xy'' - y' + x^3(e^{x^2} - p^2)y = 0$ and solve the equation.

Solution. With $z = e^{x^2/2}$ we find

$$\begin{aligned}\frac{dy}{dx} &= \frac{dz}{dx}\frac{dy}{dz} \\ &= xe^{x^2/2}\frac{dy}{dz} \qquad (9)\\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(xe^{x^2/2}\frac{dy}{dz}\right) \\ &= xe^{x^2/2}\frac{d}{dx}\left(\frac{dy}{dz}\right) + x^2e^{x^2/2}\frac{dy}{dz} + e^{x^2/2}\frac{dy}{dz} \\ &= x^2e^{x^2}\frac{d^2y}{dz^2} + x^2e^{x^2/2}\frac{dy}{dz} + e^{x^2/2}\frac{dy}{dz} \qquad (10)\end{aligned}$$

We insert this into the ODE and find

$$\begin{aligned}0 &= x^3e^{x^2}\frac{d^2y}{dz^2} + x^3e^{x^2/2}\frac{dy}{dz} + xe^{x^2/2}\frac{dy}{dz} - xe^{x^2/2}\frac{dy}{dz} + x^3(e^{x^2} - p^2)y \\ &= x^3\left(e^{x^2}\frac{d^2y}{dz^2} + e^{x^2/2}\frac{dy}{dz} + (e^{x^2} - p^2)y\right) \qquad (11)\end{aligned}$$

Dividing by x^3 we have

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - p^2)y = 0 \qquad (12)$$

We recognise this as Bessel's differential equation of order p.

In case of p being an integer the general solution is then a linear combination of the Bessel functions of first and second kind of order p, $J_p(z)$ and $Y_p(z)$ respectively (where Y_p is also called a Neumann function), i.e. expressed in terms of the original variable x we have

$$y(x) = C_1J_p(e^{x^2/2}) + C_2Y_p(e^{x^2/2}), \quad p \text{ integer.} \qquad (13)$$

If p is *not* an integer, J_p and J_{-p} are linearly independent and the general solution is instead formed by the linear combination of these (the Frobenius method then gives two linearly independent solutions) so that

$$y(x) = C_1J_p(e^{x^2/2}) + C_2J_{-p}(e^{x^2/2}), \quad p \text{ not integer.} \qquad (14)$$

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3. **P7 2010-01-04.** Use the series definition of the Bessel function,

$$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} \qquad (15)$$

to explicitly verify that

(a) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$

(b) $\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x)$

Solution. The expressions we should prove come from the general recursive relations that the Bessel functions obey, given by

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{16}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2\frac{d}{dx} J_n(x) \tag{17}$$

These can be combined to give the second relation given in the problem,

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x). \tag{18}$$

(a) From the series definition we have

$$\begin{aligned} J_1(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(1+s)!} \left(\frac{x}{2}\right)^{1+2s} \\ &= \frac{2}{x} \sum_{s=0}^{\infty} (2+s) \frac{(-1)^s}{s!(2+s)!} \left(\frac{x}{2}\right)^{2+2s} \\ &= \frac{4}{x} J_2(x) + \frac{2}{x} \sum_{s=0}^{\infty} \frac{s(-1)^s}{s!(2+s)!} \left(\frac{x}{2}\right)^{2+2s} \\ &= \frac{4}{x} J_2(x) + \sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!(2+s)!} \left(\frac{x}{2}\right)^{1+2s} \end{aligned} \tag{19}$$

where in the last equality we have used that the $s = 0$ term in the series in the rightmost term is zero so that we can start from $s = 1$ instead.

We now write down the series for J_3 and shift indices to cancel the second term in Eq. (19):

$$\begin{aligned} J_3(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(3+s)!} \left(\frac{x}{2}\right)^{3+2s} \\ &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{(s-1)!(2+s)!} \left(\frac{x}{2}\right)^{3+2(s-1)} \\ &= -\sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!(2+s)!} \left(\frac{x}{2}\right)^{1+2s}. \end{aligned} \tag{20}$$

This cancels the second term in the expression for $J_1(x)$ in Eq. (19) and we are left with

$$J_3(x) + J_1(x) = \frac{4}{x} J_2(x). \tag{21}$$

(b) We have

$$\frac{d}{dx}\left(xJ_1(x)\right) = \frac{d}{dx}\left(x\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(1+s)!}\left(\frac{x}{2}\right)^{1+2s}\right)$$
$$= 2\frac{d}{dx}\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(1+s)!}\left(\frac{x}{2}\right)^{2+2s}\right)$$
$$= 2\sum_{s=0}^{\infty}\frac{1}{2}(2s+2)\frac{(-1)^s}{s!(1+s)!}\left(\frac{x}{2}\right)^{1+2s} \tag{22}$$

where the factor 1/2 comes from the inner derivative of the expression inside the parentheses. This further simplifies to

$$\frac{d}{dx}\left(xJ_1(x)\right) = 2\sum_{s=0}^{\infty}\frac{(-1)^s}{(s!)^2}\left(\frac{x}{2}\right)^{1+2s}$$
$$= 2\frac{x}{2}\sum_{s=0}^{\infty}\frac{(-1)^s}{(s!)^2}\left(\frac{x}{2}\right)^{2s}$$
$$= xJ_0(x) \tag{23}$$

so that indeed we have

$$\frac{d}{dx}\left(xJ_1(x)\right) = xJ_0(x). \tag{24}$$

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4. **P7 2010-10-30.** The wave equation for a circular membrane with radius a is given in polar coordinates (r, φ) by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \tag{25}$$

where $r \leq a$, $0 \leq \varphi < 2\pi$, $t \geq 0$ and $u(r, \varphi, t)$ gives the displacement of the membrane. The membrane is fixed at $r = a$ which gives the boundary condition $u(a, \varphi, t) = 0$ for $0 \leq \varphi < 2\pi$ and $t \geq 0$.

(a) Find the general solution to the above problem under the assumption that we have circular symmetry, i.e. that u is independent of φ so that it can be written $u = u(r, t)$.

(b) Find the solution obtained when $a = c = 2$ and the initial conditions are $u(r, 0) = 5J_0(\alpha_3 r/2)$ and $\partial u/\partial t(r, 0) = 4\alpha_7 J_0(\alpha_7 r/2)$.

Solution. We assume in the following the physically reasonable additional condition that u is finite everywhere on the membrane, i.e. for $r \leq a$, $0 \leq \varphi < 2\pi$ and $t \geq 0$.

(a) Without φ-dependence the wave equation becomes

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}. \tag{26}$$

Assuming a separable solution we write $u(r,t) = R(r)T(t)$, insert this in the PDE and divide everywhere by $R(r)T(t)$ to obtain

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -\lambda^2 \tag{27}$$

where we have with the usual argument (of the LHS being independent of r and the RHS being independent of t) introduced a separation constant $-\lambda^2$. This separates the PDE into two ODE:s:

$$\begin{cases} \dfrac{d^2T}{dt^2} + c^2\lambda^2 T = 0 \\ r^2\dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} + \lambda^2 r^2 R = 0. \end{cases} \tag{28}$$

We see that there is (as expected for $\lambda^2 > 0$) an oscillating solution for T,

$$T(t) = A\cos(c\lambda t) + B\sin(c\lambda t). \tag{29}$$

The differential equation for $R(r)$ is a rescaled Bessel equation. To see this, write $\rho = \lambda r$ so that expressed in ρ the R equation becomes

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \rho^2 R = 0 \tag{30}$$

which we recognise as a zeroth order Bessel differential equation, with general solution given by the linear combination of zeroth order Bessel functions of first and second kind, $J_0(\rho)$ and $Y_0(\rho)$, i.e.

$$R(r) = CJ_0(\lambda r) + DY_0(\lambda r). \tag{31}$$

However, Y_0 is singular at the origin $r = 0$ and since $r = 0$ is part of the region where we want to have a finite solution we must set $D = 0$ to make sure that the solution is finite on the entire membrane. The boundary condition at $r = a$ says that the displacement there is zero, $u(a,t) = 0$, this results in the condition

$$R(a) = CJ_0(\lambda a) = 0 \quad \Rightarrow \quad \lambda a = \alpha_s,\ s = 1, 2, \ldots \tag{32}$$

where α_s is the s:th zero of J_0. Thus there is one R-solution for each s and the general solution for $u(r,t)$ is then obtained by summing all individual solutions as (absorbing all C_s in the A_s and B_s)

$$u(r,t) = \sum_{s=1}^{\infty} J_0\left(\frac{\alpha_s}{a}r\right)\left[A_s\cos\left(\frac{\alpha_s c}{a}t\right) + B_s\sin\left(\frac{\alpha_s c}{a}t\right)\right] \tag{33}$$

(b) We are now interested in the particular solution when $a = c = 2$ and the initial conditions are

$$u(r,0) = 5J_0\left(\frac{\alpha_3}{2}r\right), \qquad \frac{\partial u(r,0)}{\partial t} = 4\alpha_7 J_0\left(\frac{\alpha_7}{2}r\right). \tag{34}$$

We must then have

$$5J_0\left(\frac{\alpha_3}{2}r\right) = \sum_{s=1}^{\infty} J_0\left(\frac{\alpha_s}{2}r\right) A_s \tag{35}$$

and because the Bessel functions which scale with different zeros α_s in their arguments form a linearly independent and orthogonal set of functions[1], only the $s = 3$ term can contribute in the series on the RHS. Therefore we must have

$$A_s = \begin{cases} 5, & s = 3 \\ 0, & s \neq 3 \end{cases} \tag{36}$$

For the other initial condition we have

$$4\alpha_7 J_0\left(\frac{\alpha_7}{2}r\right) = \sum_{s=1}^{\infty} J_0\left(\frac{\alpha_7}{2}r\right) \frac{\alpha_s 2}{2} B_s = \sum_{s=1}^{\infty} J_0\left(\frac{\alpha_7}{2}r\right) \alpha_s B_s. \tag{37}$$

Again, the functions $J_0(\alpha_s r/2)$ are for different s orthogonal so that we must have

$$B_s = \begin{cases} 4, & s = 7 \\ 0, & s \neq 7 \end{cases} \tag{38}$$

and the solution is in this case given by

$$u(r,t) = 5J_0\left(\frac{\alpha_3}{2}r\right)\cos(\alpha_3 t) + 4J_0\left(\frac{\alpha_7}{2}r\right)\sin(\alpha_7 t) \tag{39}$$

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5. **P7 2011-10-29.** A vibrating circular drum membrane is fixed at the boundary $r = r_0$, i.e. $u(r_0, \varphi, t) = 0$. At time $t = 0$ the deformation of the membrane is described by the function $u(r, \varphi, 0) = f(r, \varphi)$ and the velocity $\partial u/\partial t(r, \varphi, 0) = g(r, \varphi)$. At all times the deformation of the membrane is finite. Determine the motion of the membrane by solving the associated wave equation $\nabla^2 u = (1/c^2)\partial^2 u/\partial t^2$. The normalisation of the required orthogonal functions should be considered but need not be evaluated.

[1] It is also necessary with Dirichlet or Neumann boundary conditions, which is fulfilled here. Note also that a weight r is needed for orthogonality.

Solution. With the Laplacian in cylindrical coordinates, the wave equation becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{40}$$

and we have the following boundary conditions (BC) and initial conditions (IC):

$$u(r_0, \varphi, \varphi, t) = 0, \;\; u(r, \varphi + 2\pi, t) = u(r, \varphi, t) \qquad \text{(BC)} \tag{41}$$

$$u(r, \varphi, 0) = f(r, \varphi), \quad \frac{\partial u}{\partial t}(r, \varphi, 0) = g(r, \varphi) \qquad \text{(IC)} \tag{42}$$

where we have added the periodicity condition to ensure that u is single-valued. We also assume a finite solution on the membrane, i.e. $|u(r, \varphi, t)| < \infty$ everywhere on the membrane. We begin by assuming a solution separated spatially and temporally, i.e. we assume $u(r, \varphi, t) = \psi(r, \varphi)T(t)$. We insert this in the PDE and divide throughout by $\psi(r, \varphi)T(t)$, resulting in

$$\frac{1}{\psi}\nabla^2\psi = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -k^2 \tag{43}$$

where we have introduced the separation constant $-k^2$ (with $k^2 > 0$ to get oscillating solutions in t). This gives one ODE and one PDE:

$$\begin{cases} \dfrac{d^2T}{dt^2} + k^2c^2T = 0 \\ \nabla^2\psi(r, \varphi) + k^2\psi(r, \varphi) = 0. \end{cases} \tag{44}$$

We now separate variables a second time and assume $\psi(r, \varphi) = R(r)\Phi(\varphi)$. Inserted in the PDE for ψ we then obtain after multiplying all over by $r^2/[R(r)\Phi(\varphi)]$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + k^2r^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = \ell^2 \tag{45}$$

where we have introduced a second separation constant ℓ^2 (chosen positive to obtain oscillating, periodic solutions in φ). Our set of ODE:s is then finally

$$\begin{cases} \dfrac{d^2T}{dt^2} + k^2c^2T = 0 \\ r^2\dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} + (k^2r^2 - \ell^2)R = 0 \\ \dfrac{d^2\Phi}{d\varphi^2} + \ell^2\Phi = 0 \end{cases} \tag{46}$$

The solution for $\Phi(\varphi)$ is now

$$\Phi_\ell(\varphi) = A_\ell \sin(\ell\varphi) + B_\ell \cos(\ell\varphi) \tag{47}$$

and to ensure that the solution for $u(r,\varphi,t)$ is periodic in φ we need ℓ to be an integer, so that there is one solution for every integer ℓ between $-\infty$ and $+\infty$.[2]

The differential equation for R is a Bessel differential equation of order ℓ (which is here an integer) in terms of the scaled variable $\rho = kr$ and the general solution is given by the linear combination of Bessel functions of the first and second kind since ℓ is an integer. However, Bessel functions of the second kind are singular at the origin and can therefore not be included since our solution is required to be finite everywhere on the membrane. Therefore the solution for R is (absorbing constants in A_ℓ and B_ℓ)

$$R_\ell(r) = J_\ell(kr) \tag{48}$$

and with the boundary condition at $r = r_0$ we must have

$$0 = J_\ell(kr_0) \quad \Rightarrow \quad k_{\ell m} = \frac{\alpha_{\ell m}}{r_0}, \; m = 1, 2, 3, \ldots \tag{49}$$

where $\alpha_{\ell m}$ is the m:th zero of the ℓ:th order Bessel function. Thus there is a set of linearly independent solutions for R, one for every value of ℓ and m, with each individual solution given by

$$R_{\ell m}(r) = J_\ell\left(\frac{\alpha_{\ell m}}{r_0} r\right). \tag{50}$$

Finally, the solution for T is with $k_{\ell m} = \alpha_{\ell m}/r_0$ given by

$$T_{\ell m}(t) = C_{\ell m} \sin\left(\frac{\alpha_{\ell m} c}{r_0} t\right) + D_{\ell m} \cos\left(\frac{\alpha_{\ell m} c}{r_0} t\right). \tag{51}$$

The general solution for $u(r,\varphi,t)$ is finally obtained by summing all individual solutions:

$$\begin{aligned} u(r,\varphi,t) = \sum_{\ell=-\infty}^{\infty} \sum_{m=1}^{\infty} & J_\ell\left(\frac{\alpha_{\ell m}}{r_0} r\right) \left[A_\ell \sin(\ell\varphi) + B_\ell \cos(\ell\varphi)\right] \\ & \times \left[C_{\ell m} \sin\left(\frac{\alpha_{\ell m} c}{r_0} t\right) + D_{\ell m} \cos\left(\frac{\alpha_{\ell m} c}{r_0} t\right)\right]. \end{aligned} \tag{52}$$

To obtain the particular solution of interest we lastly apply the initial conditions. At $t = 0$ we get for $u(r,\varphi,0)$

$$f(r,\varphi) = \sum_{\ell=-\infty}^{\infty} \sum_{m=1}^{\infty} J_\ell\left(\frac{\alpha_{\ell m}}{r_0} r\right) \left[A_\ell \sin(\ell\varphi) + B_\ell \cos(\ell\varphi)\right] D_{\ell m} \tag{53}$$

[2] For $\ell = 0$ only a constant solution is possible.

Using the orthogonality of sin and cos and of the J_ℓ with different zeros in the argument with weight r we have

$$\int_0^{r_0}\int_0^{2\pi} f(r,\varphi)J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right) r\sin(\ell\varphi)\; dr d\varphi = D_{\ell m}A_\ell \times N_1 \tag{54}$$
$$\int_0^{r_0}\int_0^{2\pi} f(r,\varphi)J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right) r\cos(\ell\varphi)\; dr d\varphi = D_{\ell m}B_\ell \times N_2 \tag{55}$$

which we get by multiplying by sin or cos and $J_\ell(\alpha_{\ell m}r/r_0)$ with other values of ℓ and m and integrating in r from 0 to r_0 (remembering the weight r necessary for the J_ℓ to be orthogonal) and in φ from 0 to 2π.[3] The normalisation factors are given by

$$N_1 = \int_0^{r_0}\int_0^{2\pi}\left[J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right)\right]^2 r\sin^2(\ell\varphi)\; dr d\varphi \tag{56}$$
$$N_2 = \int_0^{r_0}\int_0^{2\pi}\left[J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right)\right]^2 r\cos^2(\ell\varphi)\; dr d\varphi \tag{57}$$

The remaining constants are obtained by applying the second initial condition, for the velocity. This yields (using the same method as above, but on $\partial u/\partial t(r,\varphi,0)$)

$$\int_0^{r_0}\int_0^{2\pi} g(r,\varphi)J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right) r\sin(\ell\varphi)\; dr d\varphi = C_{\ell m}A_\ell \times \frac{\alpha_{\ell m}c}{r_0} \times N_1 \tag{58}$$
$$\int_0^{r_0}\int_0^{2\pi} g(r,\varphi)J_\ell\left(\frac{\alpha_{\ell m}}{r_0}r\right) r\cos(\ell\varphi)\; dr d\varphi = C_{\ell m}B_\ell \times \frac{\alpha_{\ell m}c}{r_0} \times N_2 \tag{59}$$

where the $(\alpha_{\ell m}c/r_0)$ comes from the time derivative. Note that the constants are all expressed in terms of *definite* integrals, i.e. they are just numbers as they should be.

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[3]Note that $D_{\ell m}A_\ell$ and $D_{\ell m}B_\ell$ should be thought of as single constants for each ℓ and m (we could have combined them into some $D_{\ell m}A_\ell = \widetilde{A}_{\ell m}$ and $D_{\ell m}B_\ell = \widetilde{B}_{\ell m}$ when writing down the general solution above). The same holds for $C_{\ell m}A_\ell = \widetilde{C}_{\ell m}$ and $C_{\ell m}B_\ell = \widetilde{D}_{\ell m}$ in the second initial condition.