Tutorial Class 5
Mathematical Methods in Physics

Carl Niblaeus

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Jordan's lemma

When using contour integrals to calculate real integrals over the real line by extending the integration into the complex plane we frequently have to consider contributions over semicircles C_R in the complex plane with radius $R \to \infty$. When we have an integrand containing a complex exponential e^{iaz} we can use *Jordan's lemma* to determine whether the integral over C_R is zero. The lemma states that the following integral goes to zero:

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\ dz = \lim_{R\to\infty} I_R = 0, \tag{1}$$

if $a > 0$, C_R is a positively oriented semicircle with radius R in the upper halfplane centered at the origin and f is such that

$$\lim_{R\to\infty} f(z) = 0 \text{ for all } z \text{ with } 0 \leq \theta \leq \pi \tag{2}$$

To prove this, write $e^{iaz} = e^{iaR\cos\theta - aR\sin\theta}$ and look at the modulus of the integral (remembering that $dz = iRe^{i\theta}d\theta$ on a circular arc of constant radius R):

$$|I_R| \leq R \int_0^\pi |f(Re^{i\theta})| e^{-aR\sin\theta}\ d\theta \tag{3}$$

Now let $|f(z)| = |f(Re^{i\theta})| < \epsilon$ for some R, then we know that $\epsilon \to 0$ when $R \to \infty$ (by assumption). Then

$$|I_R| < \epsilon R \int_0^\pi e^{-aR\sin\theta}\ d\theta = 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\ d\theta \tag{4}$$

Now use that $\sin\theta \geq 2\theta/\pi$ in the range $0 \leq \theta \leq \pi/2$. Then

$$\begin{aligned} |I_R| &< 2\epsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\ d\theta \\ &= 2\epsilon R \frac{\pi}{2aR} \left[-e^{-2aR\theta/\pi} \right]_0^{\pi/2} \\ &= \frac{\epsilon\pi}{a} \left(1 - e^{-aR}\right) \to 0 \text{ when } R \to \infty \end{aligned} \tag{5}$$

since $\epsilon \to 0$ when $R \to \infty$, i.e. $I_R = 0$ when $R \to \infty$, which completes the proof.

We can extend Jordan's lemma for the case of $a < 0$ or equivalently the case of the integral of $f(z)e^{-iaz}, a > 0$. However, then the lemma holds for a semicircle in the *lower* half-plane instead.

1. **P1 2016-01-20.** Use calculus of residues to calculate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 5}\ dx. \tag{6}$$

Solution. We denote the integral by I and note that

$$I = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 4x + 5}\ dx\right) \tag{7}$$

We now consider the contour integral

$$\oint_C f(z)\ dz = \oint_C \frac{ze^{iz}}{z^2 + 4z + 5}\ dz \tag{8}$$

where the contour C is a closed, counterclockwise semicircle in the upper halfplane with radius $R \to \infty$. Then I is equal to the imaginary part of the integral along the real line. We write

$$\oint_C f(z)\ dz = \int_{C_R} f(z)\ dz + \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 4x + 5}\ dx \tag{9}$$

and I is the imaginary part of the second integral on the right-hand side. The integrand is singular when the denominator is zero, i.e. when

$$z^2 + 4z + 5 = 0 \quad \Leftrightarrow \quad z = -2 \pm i \tag{10}$$

and the simple pole at $z = -2 + i$ is enclosed by C. Therefore, by the residue theorem we have

$$\begin{aligned} \oint_C f(z)\ dz &= 2\pi i \,\mathrm{Res}(f(z), -2 + i) \\ &= 2\pi i \lim_{z \to -2+i} (z - (-2 + i)) \frac{ze^{iz}}{(z - (-2 + i))(z - (-2 - i))} \\ &= 2\pi i \frac{(-2 + i)e^{-2i-1}}{2i} \\ &= \frac{\pi(-2 + i)}{e} (\cos 2 - i \sin 2) \\ &= \frac{\pi}{e} (\sin 2 - 2\cos 2) + i\, \frac{\pi}{e} (\cos 2 + 2 \sin 2) \end{aligned} \tag{11}$$

Jordan's lemma tells us that

$$\oint_{C_R} f(z)\ dz = 0 \tag{12}$$

when $R \to \infty$ since the integrand contains a complex exponential e^{iaz} with $a = 1 > 0$ and the remaining part of the integrand $z/(z^2 + 4z + 5)$ goes to zero when $R \to \infty$. Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 4x + 5}\, dx &= 2\pi i \operatorname{Res}(f(z), -2 + i) \\ &= \frac{\pi}{e} (\sin 2 - 2\cos 2) + i\, \frac{\pi}{e} (\cos 2 + 2\sin 2) \qquad (13)\end{aligned}$$

and

$$\begin{aligned}I &= \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 4x + 5}\, dx \right) \\ &= \frac{\pi}{e} (\cos 2 + 2\sin 2) \qquad (14)\end{aligned}$$

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2. **P1 2014-11-08.** Use calculus of residues to evaluate the integral

$$\int_0^{\infty} \frac{\cos \pi x}{x^2 + 1}\, dx. \qquad (15)$$

Specify carefully the contour used.

Try for yourself without looking at the solution!

Solution. We note first that the integrand is symmetric around $x = 0$ so that (denoting the integral we seek in Eq. (15) as I)

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2+1}\ dx = \frac{1}{2}\,\mathrm{Re}\bigg(\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{(x+i)(x-i)}\ dx\bigg) \tag{16}$$

We now consider instead the contour integral

$$\oint_C f(z)\ dz = \oint_C \frac{e^{i\pi z}}{(z+i)(z-i)}\ dz \tag{17}$$

where the contour is the closed, counterclockwise semicircle in the upper halfplane with radius R centered at zero. In the limit $R \to \infty$ the real part of the contour along the real line will yield the integral we seek, I. The (simple) poles of $f(z)$ are at $z = \pm i$ and the pole at $z = i$ lies inside our contour. Therefore by the residue theorem

$$\oint f(z)\ dz = \int_{C_R} f(z)\ dz + \int_{-\infty}^{\infty} \frac{e^{i\pi x}}{(x+i)(x-i)}\ dx \tag{18}$$

where C_R is the open halfcircle in the upper halfplane centered at zero. From Jordan's lemma we have

$$\int_{C_R} f(z)\ dz = 0 \tag{19}$$

since the exponential is e^{iaz} with $a = \pi > 0$ and the remaining part of the integrand, $1/(z^2+1)$, goes to zero for $R \to \infty$. Therefore

$$\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{(x+i)(x-i)}\ dx = 2\pi i\,\mathrm{Res}(f(z), i). \tag{20}$$

Since $z = i$ is a simple pole the right-hand side is in this case given by

$$2\pi i\,\mathrm{Res}(f(z), i). = 2\pi i \lim_{z\to i}\left[(z-i)\frac{e^{i\pi z}}{(z-i)(z+i)}\right] = 2\pi i\frac{e^{-\pi}}{2i} = \frac{\pi}{e^{\pi}} \tag{21}$$

and our integral I becomes

$$I = \frac{1}{2}\,\mathrm{Re}\Big(\frac{\pi}{e^{\pi}}\Big) = \frac{\pi}{2e^{\pi}} \tag{22}$$

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3. **P1 2015-01-02.** Specify the contour and use calculus of residues to evaluate the integral

$$\int_0^\infty \frac{\cos(x/2)}{(x^2+4)^2}\,dx. \tag{23}$$

Solution. We denote the integral by I. Note that since the integrand is an even function around $x = 0$ we can write

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x/2)}{(x^2+4)^2}\,dx = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix/2}}{(x^2+4)^2}\,dx\right) \tag{24}$$

Now consider the contour integral

$$\oint_C f(z)\,dz = \oint_C \frac{e^{iz/2}}{(z^2+4)^2}\,dz \tag{25}$$

over a contour C that is a closed, counterclockwise semicircle in the upper halfplane with radius R. The integrand is singular with double poles at $z = \pm 2i$. In the limit $R \to \infty$ the real part of the integral on the real line is equal to $2I$. By the residue theorem the whole contour integral is equal to $2\pi i$ times the sum of enclosed residues. In this case we enclose the double pole at $z = 2i$ with residue

$$\begin{aligned}\mathrm{Res}(f(z), 2i) &= \lim_{z\to 2i}\frac{d}{dz}\left[(z-2i)^2\frac{e^{iz/2}}{(z-2i)^2(z+2i)^2}\right]\\ &= \lim_{z\to 2i}\left[\frac{e^{iz/2}}{(z+2i)^2}\left(\frac{i}{2}-\frac{2}{z+2i}\right)\right]\\ &= \frac{e^{-1}}{(4i)^2}\left(\frac{i}{2}-\frac{2}{4i}\right)\\ &= -\frac{i}{16e}\end{aligned} \tag{26}$$

We split the contour in two parts: the open semicircle C_R of radius R in the upper halfplane and the integral along the real line. The integral over C_R is a case of Jordan's lemma with an exponential $e^{iz/2}$ multiplying $1/(z^2+4)^2$ (which goes to zero for $|z| \to \infty$) and is therefore zero. Then only the integral along the real line contributes to the contour integral and we have

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{-\infty}^{\infty}\frac{e^{ix/2}}{(x^2+4)^2}\,dx\\ &= 2\pi i\,\mathrm{Res}(f(z), 2i)\\ &= 2\pi i\left(-\frac{i}{16e}\right)\\ &= \frac{\pi}{8e}\end{aligned} \tag{27}$$

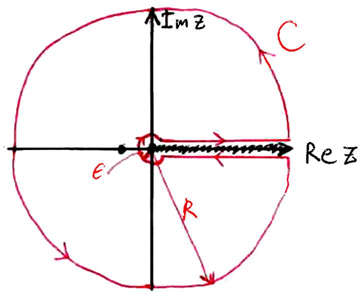

Figure 1: The contour used in P1 2013-01-03. Note that there is a branch cut along the positive real line.

So the integral we seek is

$$I = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix/2}}{(x^2+4)^2}\,dx\right) = \frac{\pi}{16e} \tag{28}$$

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4. **P1 2013-01-03.** Use calculus of residues to evaluate the integral

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx \tag{29}$$

where $0 < p < 1$.

Solution. When we extend the integration into the complex plane the integrand has a simple pole at $z = -1$ and a branch point at $z = 0$, since z^{p-1} is a multivalued function[1]. We therefore make a branch cut from the origin along the real line to infinity so that the integrand is single-valued in the whole plane. We consider then the contour integral

$$\oint_C f(z)\,dz = \oint_C \frac{z^{p-1}}{z+1}\,dz \tag{30}$$

where C is a closed contour that can *not* cross the branch cut along the positive real line. We draw the contour according to Fig. 4 and write the total contour

[1]If we follow the value of z^{p-1} on the contour the unit circle, the value will be $1^{p-1} = 1$ just when we start at $z = +1$ and $1^{p-1}e^{(p-1)2\pi i} = e^{2\pi p i} \neq 1$ after one circuit around the origin when we come back to $z = +1$.

integral as (keeping note of the counterclockwise, positive orientation)

$$\begin{aligned}\oint f(z)\ dz &= \int_{C_R} f(z)\ dz + \int_{C_\epsilon} f(z)\ dz + \int_{C_+} f(z)\ dz + \int_{C_-} f(z)\ dz \\ &= I_R + I_\epsilon + \int_\epsilon^\infty \frac{x^{p-1}}{x+1}\ dx + e^{(p-1)2\pi i}\int_\infty^\epsilon \frac{x^{p-1}}{x+1}\ dx \\ &= I_R + I_\epsilon + \left(1-e^{2\pi p i}\right)\int_\epsilon^\infty \frac{x^{p-1}}{x+1}\ dx \qquad (31)\end{aligned}$$

where we have changed the direction of integration of the integral over C_- and used that $e^{(p-1)2\pi i} = e^{2\pi p i}$. The integral in the last term is equal to the integral we seek in the limit $\epsilon \to 0$. Note the important fact that the integrals along C_+ and C_- do *not* cancel each other since the value of the integrand is different just above and below the branch cut respectively.[2]

The integrals I_R and I_ϵ are both zero in the limits $R \to \infty$, $\epsilon \to 0$. This can for example be seen by expressing the integral in polar form.

$$I_R = \int_0^{2\pi} \frac{R^{p-1}e^{(p-1)i\theta} iRe^{i\theta}}{Re^{i\theta}+1}\ d\theta = iR^p \int_0^{2\pi} \frac{e^{ip\theta}}{Re^{i\theta}+1}\ d\theta \qquad (32)$$

Therefore

$$\begin{aligned}|I_R| &\le R^p \int_0^{2\pi} \frac{d\theta}{\sqrt{1+R^2+2R\cos\theta}} \\ &\le R^p \int_0^{2\pi} \frac{d\theta}{\sqrt{1+R^2-2R}} \\ &= \frac{R^p}{(1-R)^2} 2\pi \qquad (33)\end{aligned}$$

This goes to zero when $R \to \infty$ and $0 < p < 1$. In fact it also goes to zero in the limit $R \to 0$ which is exactly the case obtained for the integral I_ϵ (the calculation is the same as for I_R except that R should be exchanged for ϵ), so both I_R and I_ϵ are zero. Then by the residue theorem we have (remember that there a simple pole at $z = -1$)

$$\begin{aligned}\left(1-e^{2\pi p i}\right)\int_\epsilon^\infty \frac{x^{p-1}}{x+1}\ dx &= 2\pi i \operatorname{Res}\left(\frac{z^{p-1}}{z+1}, -1\right) \\ &= 2\pi i \lim_{z\to -1}(z+1)\frac{z^{p-1}}{z+1} \\ &= 2\pi i \left(e^{i\pi}\right)^{p-1} \\ &= 2\pi i e^{-i\pi} e^{ip\pi} \\ &= -2\pi i e^{ip\pi} \qquad (34)\end{aligned}$$

[2]Consider z in polar coordinates. Then, just above the branch cut $z = xe^{i\cdot 0} = x$ whereas just below the branch cut, where $\theta = 2\pi$, we have $z = xe^{2\pi i}$ (using that $|z| = x$ for z along the x axis). Then the integrand has the value $(xe^0)^{p-1}/(xe^0+1) = x^{p-1}/(x+1)$ just above the branch. Just below the cut, it instead has the value $(xe^{2\pi i})^{p-1}/(xe^{2\pi i}+1) = x^{p-1}e^{2\pi p i}/(x+1)$ since $e^{2\pi i} = 1$.

Taking the limit $\epsilon \to 0$ so that our left-hand side contains the integral I we seek we get

$$\begin{aligned} I &= \frac{-2\pi i e^{ip\pi}}{(1-e^{i2p\pi})} \\ &= \frac{-2\pi i e^{ip\pi}}{e^{ip\pi}\left(e^{-ip\pi}-e^{ip\pi}\right)} \\ &= \frac{\pi}{\sin p\pi} \end{aligned} \tag{35}$$

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5. **P11.9.5.** Evaluate

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} \tag{36}$$

where a is real and not an integer.

Solution. We evaluate the sum using techniques from residue calculus. The idea is to consider the series in Eq. (**??**) as a sum of residues inside some contour C and set this sum (times $2\pi i$) equal to the contour integral around C of the specific function which fulfils that. To do this, we first note that for n integer, the function $\pi \csc(\pi z) = \pi / \sin(\pi z)$ has simple poles at $z = n$ and we find with the aid of l'Hôpital's rule that the residues there are

$$\begin{aligned} \operatorname{Res}(\pi \csc(\pi z), z=n) &= \lim_{z\to n}(z-n)\frac{\pi}{\sin(\pi z)} \\ &= \lim_{z\to n}\frac{1}{\cos(\pi z)} \\ &= (-1)^n. \end{aligned} \tag{37}$$

since $\cos(n\pi) = (-1)^n$. We then consider the contour integral of $f(z)\pi\csc(\pi z)$, where $f(z)$ is a meromorphic function[3], around a large counterclockwise circle C of radius $N + \frac{1}{2}$, centered at the origin. We have then from the residue theorem

$$\begin{aligned} \oint_C f(z)\pi\csc(\pi z)\, dz = 2\pi i\Bigg[&\sum_{n=-N}^{N} (-1)^n f(n) \\ &+ \sum_i (\text{residues of } f(z)\pi\csc(\pi z) \text{ at singularites } z_i \text{ of } f(z))\Bigg] \end{aligned} \tag{38}$$

[3]A meromorphic function is a function that is analytic in its domain except for possibly a discrete set of finite-order, isolated poles but no essential singularities.

and if $zf(z) \to 0$ when $|z| \to \infty$ the entire contour integral vanishes when we let $N \to \infty$ so that

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_i \left(\text{residues of } f(z)\pi \csc(\pi z) \text{ at singularites } z_i \text{ of } f(z)\right) \tag{39}$$

where we assume that f has singularities only at non-integer z. That is, we can evaluate the series $\sum_{n=-\infty}^{\infty}(-1)^n f(n)$ for some f that is non-singular at all n by summing up all the residues of the expression $f(z)\pi \csc(\pi z)$ at the points where $f(z)$ is singular. In this case we have to choose $f(z) = 1/(z+a)^2$ which has one singularity in the complex plane: a double pole at $z = -a$ where a is a non-integer. The residue of $f(z)\pi \csc(\pi z)$ is there given by

$$\begin{aligned}
\text{Res}(f(z)\pi \csc(\pi z), z = -a) &= \lim_{z \to -a} \frac{1}{(2-1)!} \frac{d}{dz} \left[(z+a)^2 \frac{\pi \csc(\pi z)}{(z+a)^2} \right] \\
&= \pi \lim_{z \to -a} \frac{d}{dz} \csc(\pi z) \\
&= \pi \lim_{z \to -a} \frac{(-\pi)\cos(\pi z)}{\sin^2(\pi z)} \\
&= -\pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)}
\end{aligned} \tag{40}$$

where we note that a is *not* an integer so that the expression is finite. The sum then becomes

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)} \tag{41}$$
