Tutorial Class 4
Mathematical Methods in Physics

Carl Niblaeus

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The residue theorem

There are many aspects of complex analysis covered in this course in chapter 11. In my opinion the most important thing to remember is the *residue theorem*. The theorem states

$$\oint_C f(z)\ dz = 2\pi i \sum_i \mathrm{Res}(f(z), z_i) \tag{1}$$

where $\mathrm{Res}(f(z), z_i)$ is the *residue* (the coefficient of $1/z$ in the Laurent expansion of f) of the complex-valued function $f(z)$ at the pole z_i, which is in the interior of the closed contour C. The meaning of the theorem is that we can find the value of a contour integral of a complex-valued function over a closed contour by computing the residues at its singularities inside the contour. We can also use this to evaluate real-valued integrals by extending the integration into the complex plane. The residue of an n:th order pole at z_0 is calculated as

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right] \tag{2}$$

Notably a simple pole (with $n = 1$) has residue $\lim_{z \to z_0}(z - z_0) f(z)$.

A very important property of contour integrals is the fact that the integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour within the region of analyticity.

Starting with the residue theorem we can obtain some well known special cases, for example *Cauchy's integral formula*, which states that

$$\frac{1}{2\pi i} \oint \frac{f(z)\ dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \text{ within the contour,} \\ 0, & z_0 \text{ exterior to the contour} \end{cases} \tag{3}$$

where $f(z)$ here needs to be analytic in the whole interior of C (unlike in the residue theorem above!) and $f(z_0)$ is exactly the residue of $f(z)/(z-z_0)$ at z_0. The case where z_0 lies outside the contour yields the *Cauchy integral theorem* which states that the contour integral over a closed curve is zero if the integrand $f(z)$ is analytic within the whole interior of C.

1. **P11.2.11.** Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x,y) + i\,v(x,y)$. We label the real part $u(x,y)$ the velocity potential and the imaginary part $v(x,y)$ the stream function. The fluid velocity is given by $\vec{V} = \vec{\nabla} u$. If $f(z)$ is analytic:

   (a) Show that $df/dz = V_x - iV_y$.

   (b) Show that $\vec{\nabla} \cdot \vec{V} = 0$ (no sources or sinks).

   (c) Show that $\vec{\nabla} \times \vec{V} = 0$ (irrotational, nonturbulent flow).

   Solution. First of all, note that

$$\vec{V} = (V_x, V_y) = \vec{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \tag{4}$$

   Second of all, note that if f is analytic its real and imaginary parts obey the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5}$$

   (a) We have

$$\begin{aligned} \frac{df}{dz} &= \frac{du}{dz} + i\frac{dv}{dz} \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} \right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z} \right) \end{aligned} \tag{6}$$

   where we have used the chain rule. Using that

$$x = \frac{z+z^*}{2}, \quad y = \frac{z-z^*}{2i} \tag{7}$$

   we find then

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x}\frac{1}{2} + \frac{\partial u}{\partial y}\frac{1}{2i} + i\frac{\partial v}{\partial x}\frac{1}{2} + i\frac{\partial v}{\partial y}\frac{1}{2i} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ &= \frac{1}{2}\left(2\frac{\partial u}{\partial x} \right) + \frac{i}{2}\left(-2\frac{\partial u}{\partial y} \right) \\ &= V_x - iV_y \end{aligned} \tag{8}$$

   where we have used the Cauchy–Riemann equations in the next to last step.

   (b) The divergence of $\vec{V}$ is

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} \right) \tag{9}$$

We now use the Cauchy–Riemann equations to exchange $\partial u/\partial x$ for $\partial v/\partial y$ in the first term and $\partial u/\partial y$ for $-\partial v/\partial x$ and then exchange the order of the derivatives to obtain

$$\begin{aligned} \vec{\nabla}\cdot\vec{V} &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= 0 \end{aligned} \tag{10}$$

(c) The curl of $\vec{V}$ is

$$\begin{aligned} \vec{\nabla}\times\vec{V} &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix} \\ &= \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \\ &= \frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial u}{\partial x} \\ &= 0 \end{aligned} \tag{11}$$

where we have changed the order of the derivatives in one of the terms in the last step.

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2. **P11.3.7.** Show that

$$\oint_C \frac{dz}{z^2+z} = 0 \tag{12}$$

in which the contour C is a circle defined by $|z| = R > 1$.
Hint: Direct use of the Cauchy integral theorem is illegal. The integral may be evaluated by expanding into partial fractions and then treating the two terms individually. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

Solution. The contour C is shown in the left part of Fig. 1. As stated we can not use the Cauchy integral theorem (which states that $\oint_C f(z)\,dz = 0$ for f analytic inside C) directly since the integrand has poles at $z = 0$ and $z = -1$ and hence is not analytic inside C. We do a partial fraction decomposition,

$$\frac{1}{z^2+z} = \frac{A}{z} + \frac{B}{z+1} = \frac{A+(A+B)z}{z^2+z} \tag{13}$$

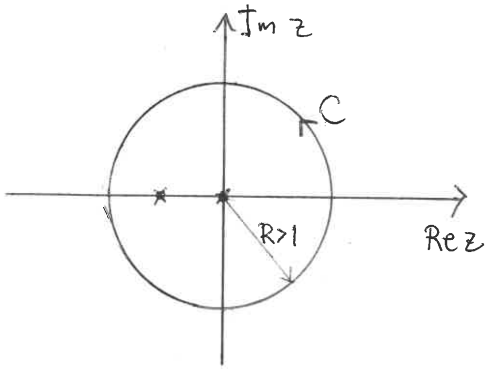

Figure 1: The contours and the positions of the poles in P11.3.7.

for some constants A and B. Comparing the LHS and RHS we see that we have $A = 1$ and $B = -A = -1$ and

$$\frac{1}{z^2+z} = \frac{1}{z} - \frac{1}{z+1} \tag{14}$$

so that the integral is

$$\oint_C \frac{dz}{z^2+z} = \oint_C \frac{dz}{z} - \oint_C \frac{dz}{z+1}. \tag{15}$$

The integrand in the first of these integrals can be written as $f(z)/(z-0)$ with $f(z) = 1$, i.e. f is analytic within all of C. Therefore the first integral is equal to $2\pi i$ times $f(0) = 1$. The same reasoning for the second integral with integrand $f(z)/(z-1)$ with $f(z) = 1$ analytic within C gives that the second integral is $2\pi i \cdot 1$. In total we find

$$\oint_C \frac{dz}{z^2+z} = 2\pi \cdot 1 - 2\pi i \cdot 1 = 0 \tag{16}$$

We could also have used the residue theorem directly, this would have resulted in residues

$$\mathrm{Res}\left(\frac{1}{z^2+z}, 0\right) = \lim_{z\to 0}(z-0)\frac{1}{z^2+z} = \lim_{z\to 0}\frac{1}{z+1} = 1 \tag{17}$$

$$\mathrm{Res}\left(\frac{1}{z^2+z}, -1\right) = \lim_{z\to -1}(z-(-1))\frac{1}{z^2+z} = \lim_{z\to 0}\frac{1}{z} = -1 \tag{18}$$

giving a zero value for the integral:

$$\oint_C \frac{dz}{z^2+z} = 2\pi i\,(1+(-1)) = 0 \tag{19}$$

3. **P11.4.1** Show that

$$\frac{1}{2\pi i} \oint z^{m-n-1} \, dz = \delta_{mn} \tag{20}$$

where δ_{mn} is the Kronecker delta with m and n integers and the contour encircles the origin once.

Solution. An earlier result in the book (Eq. 11.29) is that for a counterclockwise closed path C that encloses z_0 we have for any integer n

$$\oint_C (z - z_0)^n \, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases} \tag{21}$$

which can easily be calculated (see below). Using this, we have for the contour integral in this problem

$$\begin{aligned} \frac{1}{2\pi i} \oint z^{m-n-1} \, dz &= \frac{1}{2\pi i} \begin{cases} 0, & m-n-1 \neq -1 \\ 2\pi i, & m-n-1 = -1 \end{cases} \\ &= \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \\ &= \delta_{mn} \end{aligned} \tag{22}$$

Calculating the integral in Eq. (21)**:** Taking the contour C to be a circle with unit radius centered around z_0 we can express z as $z = z_0 + \rho e^{i\theta} = z_0 + e^{i\theta}$ where $\rho = 1$ and θ are the radius and angle as measured with respect to the point z_0. We then have $dz = ie^{i\theta} d\theta$ and we can write the integral as

$$\oint_C (z - z_0)^n \, dz = \int_0^{2\pi} e^{ni\theta} ie^{i\theta} \, d\theta \tag{23}$$

For $n = -1$ we get

$$\oint_C (z - z_0)^n \, dz = i \int_0^{2\pi} 1 \, d\theta = 2\pi i \quad (n = -1) \tag{24}$$

whereas for $n \neq -1$ we get

$$\oint_C (z - z_0)^n \, dz = i \int_0^{2\pi} e^{(n+1)i\theta} \, d\theta = \frac{i}{i(n+1)} \left[e^{(n+1)i\theta} \right]_0^{2\pi} = 0 \quad (n \neq -1) \tag{25}$$

since $e^{(n+1)i2\pi} = e^0 = 1$ for integer n.

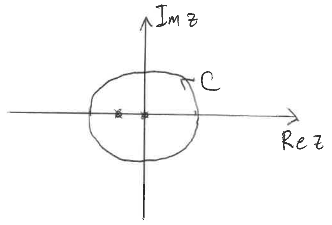

Figure 2: The contour and the position of the poles in P11.4.8.

4. **P11.4.8.** Evaluate

$$\oint_C \frac{dz}{z(2z+1)}, \tag{26}$$

for the contour the unit circle.

Solution. This problem can be solved in the same way as P11.3.7, by making a partial fraction decomposition of the integrand, deforming the contour into two circles C_1 and C_2, each encircling one of the two poles at $z = 0$ and $z = -1/2$ and noting that only one of the terms contributes for each C_i integral with values according the Cauchy's integral formula, and the results cancel out to give zero.

We can also do it by applying the residue theorem (which is just a consequence of Cauchy's integral formula with the integrand expanded as a Laurent series). To find the poles of the integrand we solve

$$z(2z+1) = 0 \quad \Rightarrow \quad z = 0,\ z = -\frac{1}{2} \tag{27}$$

and thus the poles of the integrand are found at $z = 0$ and $z = -1/2$, both inside the contour (see Fig. 2). The respective residues are given by

$$\mathrm{Res}\left(\frac{1}{z(2z+1)}, 0\right) = \lim_{z\to 0}(z-0)\frac{1}{z(2z+1)} = \lim_{z\to 0}\frac{1}{2z+1} = 1 \tag{28}$$

$$\mathrm{Res}\left(\frac{1}{z(2z+1)}, -\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}(z - \left(-\frac{1}{2}\right))\frac{1}{z(2z+1)} = \lim_{z\to -\frac{1}{2}}\frac{1}{2z} = -1 \tag{29}$$

and therefore the value of the contour integral is

$$\begin{aligned}\oint_C \frac{dz}{z(2z+1)} &= 2\pi i\left[\mathrm{Res}\left(\frac{1}{z(2z+1)}, 0\right) + \mathrm{Res}\left(\frac{1}{z(2z+1)}, -\frac{1}{2}\right)\right] \\ &= 2\pi i(1 + (-1)) \\ &= 0 \end{aligned} \tag{30}$$
