Tutorial Class 3
Mathematical Methods in Physics

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Green's functions – some general remarks

There are a number of properties of Green's functions that we will use several times, we list these here in order to be able to refer to them later.

For an ODE corresponding to a Hermitian operator $\mathcal{L}$ acting on y on some interval (a, b) so that

$$\mathcal{L}y = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x), \quad y(a) = 0,\ y(b) = 0 \tag{1}$$

we can find the solution y by first finding the *Green's function* $G(x, t)$ and then convolving $G(x, t)$ with $f(x)$ to find a solution y:

$$y(x) = \int_a^b G(x, t)f(t)\ dt \tag{2}$$

where $x, t \in (a, b)$. This works because the Green's function solves the ODE

$$\mathcal{L}G(x, t) = \delta(x - t) \tag{3}$$

so that

$$\mathcal{L}y = \int_a^b \mathcal{L}G(x, t)f(t)\ dt = \int_a^b \delta(x - t)f(t)\ dt = f(x). \tag{4}$$

Note that this is the unique solution to our ODE, since the boundary conditions are taken care of in the construction of G.

Property 1. The first property we list is that $G(x, t)$ is a solution to the *homogeneous* ODE with $f(x) = 0$ whenever $x \neq t$ since the delta function is then zero. Therefore

$$\begin{aligned} G(x, t) &= \begin{cases} h_1(t)y_1(x), & x < t \\ h_2(t)y_2(x), & x > t \end{cases} \\ &\equiv \begin{cases} G_-(x, t), & x < t \\ G_+(x, t), & x > t \end{cases} \end{aligned} \tag{5}$$

where $y_i(x)$ are solutions of the homogeneous equation that obey the boundary conditions, $y_1(a) = 0$, $y_2(b) = 0$, and h_i are functions of t.

Property 2. $G(x, t)$ is *continuous* at $x = t$,

$$G_+(x = t, t) = G_-(x = t, t) \tag{6}$$

Property 3. The x-derivative of $G(x, t)$ (which we will denote $G'(x, t)$) has a *jump discontinuity* at $x = t$ (this generates the delta function on the RHS in the ODE for $G(x, t)$):

$$\frac{\partial G_+(x = t, t)}{\partial x} - \frac{\partial G_-(x = t, t)}{\partial x} = \frac{1}{p(t)} \tag{7}$$

For an operator which is not hermitian, the RHS of this may or may not be equal to $1/p(t)$. The RHS value can be found in general by integrating the defining equation of $G(x, t)$, $\mathcal{L}G(x, t) = \delta(x - t)$, over a small interval between $t - \epsilon$ and $t + \epsilon$ and taking the limit $\epsilon \to 0$, typically using integration by parts to relate the integrals containing G'', G' and G respectively and using the fact that $G(x, t)$ is continuous at $x = t$.

Note: We have required *homogeneous* boundary conditions ($y(a) = y(b) = 0$) to ensure that $\mathcal{L}$ is hermitian. Green's functions are not exclusively defined for hermitian operators, in fact every linear differential operator admits a Green's function and we can use the Green's function technique also for problems with *inhomogeneous* boundary conditions, if $y(a) = c_1$, $y(b) = c_2$, we should then express our problem in terms of

$$u = y - \frac{c_1(b - x) + c_2(x - a)}{b - a} \tag{8}$$

where then $u(a) = u(b) = 0$. We can also use Green's functions for *initial value problems* (which also are not Sturm-Liouville problems with a hermitian operator) where y and y' are defined at one point (for example at $x = 0$) but there is no upper boundary.

1. **P10.1.4.** Find the Green's function corresponding to

$$-\frac{d^2y}{dx^2} - \frac{y}{4} = f(x) \tag{9}$$

with boundary conditions $y(0) = y(\pi) = 0$.

Solution. We begin by finding the solution of the homogeneous ODE with $f(x) = 0$. This will determine the x-dependence of $G(x, t)$. The homogenous equation is

$$-\frac{d^2y}{dx^2} - \frac{y}{4} = \frac{d}{dx}\left((-1)\frac{dy}{dx}\right) - \frac{1}{4}y = 0 \tag{10}$$

i.e. $p(x) = -1$. This ODE has linearly independent solutions $y_1 = \sin(x/2)$ and $y_2 = \cos(x/2)$. The boundary condition at $x = 0$ is satisfied by y_1 and the boundary condition at $x = \pi$ is satisfied by y_2. We thus have in accordance with *Property 1*

$$G_-(x,t) = h_1(t)y_1(x),\ x < t, \tag{11}$$
$$G_+(x,t) = h_2(t)y_2(x),\ x > t \tag{12}$$

Property 2 says that $G_-(t,t) = G_+(t,t)$ i.e.

$$\begin{aligned} &h_1(t)\sin(t/2) = h_2(t)\cos(t/2) \\ \Rightarrow\quad &h_2(t) = h_1(t)\tan(t/2) \end{aligned} \tag{13}$$

Property 3 says that there is a discontinuity in $G'(x,t)$ (prime denoting x-derivative) at $x = t$, we find (using the relation between h_1 and h_2 above)

$$\begin{aligned} &h_2(t)\left(-\frac{1}{2}\sin(t/2)\right) - h_1(t)\frac{1}{2}\cos(t/2) = \frac{1}{-1} \\ \Rightarrow\quad &h_1(t)\tan(t/2)\sin(t/2) + h_1(t)\cos(t/2) = 2 \\ \Rightarrow\quad &h_1(t) = \frac{2}{\cos(t/2) + \sin(t/2)\tan(t/2)} = 2\cos(t/2) \end{aligned} \tag{14}$$

resulting in $h_2(t) = 2\sin(t/2)$ and $G(x,t)$ is

$$G(x,t) = \begin{cases} 2\cos(t/2)\sin(x/2), & x < t \\ 2\sin(t/2)\cos(x/2), & x > t \end{cases} \tag{15}$$

and we see that $G(x,t)$ is symmetric in the sense that $G_+(x,t) = G_-(t,x)$ which is true for hermitian operators. The solution to the inhomogeneous ODE can now be found according to

$$y(x) = \int_0^\pi G(x,t)f(t)\ dt. \tag{16}$$

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2. **P7, 2014-11-08.** Determine the Green's function for the differential equation

$$\left(\frac{d^2}{dx^2} - \lambda^2\right) y(x) = R(x) \tag{17}$$

for a positive constant λ on the interval $(-\infty, \infty)$ where the boundary conditions are that $y(-\infty) = y(\infty) = 0$.

Try this problem yourself first without looking at the solution.

Solution. The independent solutions to the homogeneous ODE with $R(x) = 0$,

$$\frac{d^2y}{dx^2} - \lambda^2 y = \frac{d}{dx}\left(1\frac{dy}{dx}\right) - \lambda^2 y = 0 \tag{18}$$

(i.e. $p(x) = 1$) are in this case

$$y_1 = e^{\lambda x}, \quad y_2 = e^{-\lambda x} \tag{19}$$

and the general solution is a sum of these with arbitrary coefficients. We note that for a t dividing the real line into the intervals $(-\infty, t)$ and (t, ∞), only y_1 satisfies the boundary condition at $-\infty$ (y_2 diverges for x large and negative), whereas only y_2 satisfies the boundary condition at $+\infty$ (y_1 diverges for x large and positive) and according to *Property 1* the Green's function can therefore be written

$$G(x,t) = \begin{cases} h_1(t)e^{\lambda x}, & x < t \\ h_2(t)e^{-\lambda x}, & x > t \end{cases} \tag{20}$$

where we will write $G_-(x,t)$ for the case $x < t$ and $G_+(x,t)$ for the case $x > t$. Using *Property 2* we find that

$$h_1(t)e^{\lambda t} = h_2(t)e^{-\lambda t} \quad \Rightarrow \quad h_2(t) = h_1(t)e^{2\lambda t}. \tag{21}$$

Putting this into the expression obtained by using *Property 3* we then find

$$h_2(t)(-\lambda)e^{-\lambda t} - h_1(t)\lambda e^{\lambda t} = 1$$

$$\Rightarrow h_1(t) = -\frac{e^{-\lambda t}}{2\lambda} \tag{22}$$

$$\Rightarrow h_2(t) = -\frac{e^{\lambda t}}{2\lambda} \tag{23}$$

and the Green's function becomes

$$G(x,t) = -\frac{1}{2\lambda}\begin{cases} e^{\lambda(x-t)}, & x < t \\ e^{\lambda(t-x)}, & x > t \end{cases} \tag{24}$$

which is symmetric since this is a Sturm-Liouville problem, i.e. $G_+(x,t) = G_-(t,x)*$. Note that we can also write the $G(x,t)$ for all x as

$$G(x,t) = -\frac{1}{2\lambda}e^{-|t-x|}. \tag{25}$$

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3. **P9, 2014-01-02.** Determine the Green's function and then use it to solve the following initial value problem:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = \sin \omega t, \quad y(0) = y'(0) = 0 \tag{26}$$

Verify that your solution satisfies the initial conditions.

Solution. This is an *initial value problem* and not a Sturm-Liouville eigenvalue problem. We can still use the same technique to construct a Green's function $G(t, s)$ with t, s in the interval $[0, \infty)$ however, but the symmetry property of the solution will be lost. Here, the solutions to the homogeneous ODE are of the form e^{mt} where

$$m^2 + 3m - 4 = 0 \quad \Leftrightarrow \quad m = -4, 1 \tag{27}$$

i.e.

$$y_1 = e^{-4t}, \quad y_2 = e^t \tag{28}$$

A general solution is then given by

$$y = c_1 e^{-4t} + c_2 e^t \tag{29}$$

and the initial values of y and y' require

$$y(0) = c_1 + c_2 = 0 \tag{30}$$

$$y'(0) = -4c_1 + c_2 = 0 \tag{31}$$

and only the trivial solution $c_1 = c_2 = 0$ satisfies this. Therefore $G(t, s) = 0$ for $t < s$. For $t > s$ we have no initial or similar conditions and $G(t, s)$ is some linear combination of y_1 and y_2 with s-dependent coefficients,

$$G(t, s) = h_1(s)e^{-4t} + h_2(s)e^t, \quad t > s. \tag{32}$$

We now use the two constraints *Property 2* and *Property 3* to determine the two unknowns h_1 and h_2. *Property 2* requires continuity in G at $t = s$, and since the solution for $t < s$ is $G = 0$ we have

$$h_1(s)e^{-4s} + h_2(s)e^s = 0 \quad \Rightarrow \quad h_2(s) = -h_1(s)e^{-5s}. \tag{33}$$

Property 3 requires the t-derivative of G to be discontinuous at $t = s$, we find

$$-4h_1(s)e^{-4s} + h_2(s)e^s - 0 = 1$$

$$\Rightarrow \quad -4h_1(s)e^{-4s} - h_1(s)e^{-5s}e^s = 1$$

$$\Rightarrow \quad h_1(s) = -\frac{1}{5}e^{4s} \tag{34}$$

$$\Rightarrow \quad h_2(s) = \frac{1}{5}e^{-s} \tag{35}$$

so the Green's function is

$$
\begin{aligned}
G(t,s) &= \begin{cases} 0, & t < s \\ -\dfrac{e^{4s}}{5}e^{-4t} + \dfrac{e^{-s}}{5}e^{t}, & t > s \end{cases} \\
&= \begin{cases} 0, & t < s \\ \dfrac{1}{5}e^{t-s}\left(1 - e^{-5(t-s)}\right), & t > s \end{cases}
\end{aligned} \tag{36}
$$

which is *not* symmetric as the Green's functions in the previous problems were. To solve the initial value problem with a RHS of $\sin\omega t$ we use the fact that

$$
\begin{aligned}
y(t) &= \int_0^\infty G(t,s)\sin\omega s \; ds \\
&= \int_0^t G(t,s)\sin\omega s \; ds
\end{aligned} \tag{37}
$$

since $G(t,s) = 0$ for $s > t$. Inserting our Green's function we find

$$
y(t) = \frac{1}{5}\int_0^t e^{t-s}\left(1 - e^{-5(t-s)}\right)\sin\omega s \; ds \tag{38}
$$

and using[1]

$$
\int_0^t e^{a(t-s)}\sin\omega s \; ds = \frac{\omega e^{at} - \omega\cos\omega t - a\sin\omega t}{\omega^2 + a^2} \tag{39}
$$

we have

$$
y(t) = \frac{1}{5}\left\{\frac{\omega e^{t} - \omega\cos\omega t - \sin\omega t}{\omega^2+1} - \frac{\omega e^{-4t} - \omega\cos\omega t + 4\sin\omega t}{\omega^2+16}\right\} \tag{40}
$$

Last of all, we verify that this satisfies the initial conditions $y(0) = y'(0) = 0$:

$$
y(0) = \frac{1}{5}\left\{\frac{\omega e^{0} - \omega\cos 0 - \sin 0}{\omega^2+1} - \frac{\omega e^{0} - \omega\cos 0 + 4\sin 0}{\omega^2+16}\right\} = 0 \tag{41}
$$

$$
y'(0) = \frac{1}{5}\left\{\frac{\omega e^{0} + \omega^2\sin 0 - \omega\cos 0}{\omega^2+1} - \frac{-4\omega e^{0} + \omega^2\sin 0 + 4\omega\cos 0}{\omega^2+16}\right\} = 0 \tag{42}
$$

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4. **P5, 2006-06-02** Use the Green's functions technique to determine the function y satisfying the differential equation

$$
\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} - \frac{4}{x^2}y = f(x) \tag{43}
$$

[1] The integral I can for example be found by integrating by parts twice. Doing that, we find that I is equal to two boundary terms plus I itself multiplied by a constant and this relation can be solved for I.

over the interval $[0, \infty)$ under the condition that y is everywhere finite on this interval and $f(x)$ is given by

$$f(x) = \begin{cases} \exp(-\lambda x), & 1 \leq x < \infty \\ 0, & 0 \leq x < 1 \end{cases} \tag{44}$$

Solution. The differential operator is not hermitian in this case but we can find the Green's function by applying the same conditions as before. The Green's function satisfies

$$\frac{d^2G(x,t)}{dx^2} + \frac{4}{x}\frac{dG(x,t)}{dx} - \frac{4}{x^2}G(x,t) = \delta(x-t) \tag{45}$$

and its x-dependence is given by the solutions of the homogeneous ODE with RHS equal to zero for $x \neq t$. To find the homogeneous solutions we try the substitution $x = e^z$ and solve the ODE in terms of z first. We did this substitution in the last tutorial class and found then (see solutions to tutorial class 2)

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \tag{46}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dz}\right) \tag{47}$$

Inserting these in the ODE we find that expressed in terms of z it is given by $y'' + 3y' - 4y = 0$. This has solutions $y_i \sim e^{m_i z}$ where

$$m^2 + 3m - 4 = 0 \quad \Rightarrow \quad m = -4, 1 \tag{48}$$

i.e.

$$y_1 = e^{-4z} = \frac{1}{x^4}, \quad y_2 = e^z = x. \tag{49}$$

The solutions can also be found by assuming the ansatz $y \sim x^\alpha$ and solving the resulting algebraic equation for α, resulting in $\alpha = -4, 1$. The boundary conditions require that y is finite everywhere on $[0, \infty)$. Dividing the interval into $[0, t)$ and (t, ∞) we see that for $x < t$, only y_2 is finite on the entire interval $[0, t)$ and for $x > t$ only y_1 is finite on all of (t, ∞). Therefore we express the Green's function in accordance with *Property 1* as

$$G(x,t) = \begin{cases} h_1(t)x, & x < t \\ h_2(t)\frac{1}{x^4}, & x > t \end{cases} \tag{50}$$

To find the t-dependence (i.e. the functions h_1 and h_2) we again use *Property 2* and *Property 3*. With *Property 2* we obtain

$$h_1(t)t = h_2(t)\frac{1}{t^4} \quad \Rightarrow \quad h_2(t) = h_1(t)t^5 \tag{51}$$

and *Property 3* results in[2]

$$h_2(t)\left(-\frac{4}{t^5}\right) - h_1(t) = 1 \tag{52}$$

This results in the Green's function being given by

$$G(x,t) = -\frac{1}{5}\begin{cases} x, & x < t \\ t^5\dfrac{1}{x^4}, & x > t \end{cases} \tag{53}$$

and we note the lack of symmetry which comes from the fact that $\mathcal{L}$ is not hermitian. The solution for y is obtained by convolving $G(x,t)$ with $f(x)$

$$y(x) = \int_0^\infty G(x,t)f(t)\ dt = -\frac{1}{5x^4}\int_1^x t^5 e^{-\lambda t}\ dt - \frac{x}{5}\int_x^\infty e^{-\lambda t}\ dt. \tag{54}$$

This can be solved by integrating the first integral multiple times using integrations by parts. This results in

$$\begin{aligned} y(x) = \frac{1}{5\lambda^6 x^4}\Big\{&e^{-\lambda x}\left[5\lambda^4x^4 + 20\lambda^3x^3 + 60\lambda^2x^2 + 120\lambda x + 120\right] \\ &- e^{-\lambda}\left[\lambda^5 + 5\lambda^4 + 20\lambda^3 + 60\lambda^2 + 120\lambda + 120\right]\Big\}. \end{aligned} \tag{55}$$

Alternative solution. We can also solve this problem by multiplying the ODE by an integrating factor

$$e^{\int(4/x)\ dx} = e^{4\ln x} = x^4 \tag{56}$$

and writing it in a self-adjoint Sturm-Liouville form as

$$x^4y'' + 4x^3y' - 4x^2y = \left(x^4y'\right)' - 4x^2y = x^4f(x). \tag{57}$$

This results in a different Green's function for the self-adjoint operator $x^4\mathcal{L}$, given by

$$\widetilde{G}(x,t) = -\frac{1}{5}\begin{cases} \dfrac{1}{t^4}x, & x < t \\ t\dfrac{1}{x^4}, & x > t \end{cases} \tag{58}$$

which *is* symmetric, as expected for a hermitian operator. The solution for y is however still the same as before since the RHS in the inhomogeneous equation is now $x^4f(x)$ rather than just $f(x)$,

$$\begin{aligned} y(x) = \int_0^\infty \widetilde{G}(x,t)\,t^4f(t)\ dt &= -\frac{1}{5x^4}\int_1^x t\,t^4e^{-\lambda t}\ dt - \frac{x}{5}\int_x^\infty \frac{1}{t^4}t^4e^{-\lambda t}\ dt \\ &= -\frac{1}{5x^4}\int_1^x t^5e^{-\lambda t}\ dt - \frac{x}{5}\int_x^\infty e^{-\lambda t}\ dt. \end{aligned} \tag{59}$$

[2] We need to be a bit careful with the RHS of the discontinuity requirement in this case, since we're not dealing with a self-adjoint operator and hence don't have a specified $p(t)$ to put in the denominator. The proper way to do it then is to integrate the defining equation for the Green's function $G'' + r(x)G' + s(x)G = \delta(x-t)$ over a small interval $[t-\epsilon, t+\epsilon]$ and take the limit $\epsilon \to 0$. In this case it will give the condition $\lim_{\epsilon\to 0}(G'(t+\epsilon,t) - G'(t-\epsilon,t) = 1$.