Tutorial Class 2
Mathematical Methods in Physics

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1. **P5, exam 2015-01-02** Solve the following differential equation:

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4\,y = 6\,x^2 \ln x. \tag{1}$$

(**Hint:** The substitution $x = e^z$ can be helpful.)

Solution. This is an *inhomogeneous* ODE of second order since y does not occur to the same power in all terms. The general solution is then

$$y = C_1 y_1 + C_2 y_2 + y_p \tag{2}$$

where y_1, y_2 are the two solutions to the homogeneous equation (with the RHS equal to zero) and y_p a *particular solution* of the inhomogeneous ODE. We begin by making the suggested change of variables $x = e^z$ which leads to:

$$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dz} = e^{-z}\frac{dy}{dz} \tag{3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right) = \frac{1}{x}\frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right) = \frac{1}{x}\left(-e^{-z}\frac{dy}{dz} + e^{-z}\frac{d^2y}{dz^2}\right) =$$
$$= \frac{1}{x^2}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right) \tag{4}$$

Inserting this into the ODE results in

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 6ze^{2z} \equiv F(z) \tag{5}$$

We will find the solution expressed in terms of z and change back to x in the end. We begin by solving the homogeneous ODE given by $y'' - 4y' + 4y = 0$. This is a second order ODE with constant coefficients and so has solutions of form e^{mz} where m are roots of the equation obtained by inserting e^{mz} in the ODE:

$$m^2 - 4m + 4 = 0 \tag{6}$$
$$\Leftrightarrow (m-2)^2 = 0 \tag{7}$$

i.e. $m = 2$ is a double root. In the case of a double root $m = 2$, the two linearly independent solutions[1] are

$$y_1 = e^{mz} = e^{2z}, \quad y_2 = ze^{mz} = ze^{2z}. \tag{8}$$

We must now find the particular solution y_p. Following the discussion on p. 375 in the book (AWH) (this is called the method of *variation of parameters*) we write

$$y_p(z) = u_1(z)y_1(z) + u_2(z)y_2(z) \tag{9}$$

where the u_i are functions of z, therefore not restricting the functional form of y_p. The derivative is

$$y_p' = u_1 y_1' + u_2 y_2' + (y_1 u_1' + y_2 u_2') \tag{10}$$

and we can assume that the expression in brackets vanishes without inconsistency.[2] Using this assumption we take the second derivative and insert into the ODE, this leads to a system of equations for u_1' and u_2' (the first line is our assumption):

$$y_1 u_1' + y_2 u_2' = 0 \tag{11}$$

$$y_1' u_1' + y_2' u_2' = F(z) \tag{12}$$

The determinant of this system is the Wronskian, which is non-zero since y_1 and y_2 are independent, hence we have a unique solution to the system which justifies our assumption made earlier. We solve it by inserting $u_1' = -(y_2/y_1)u_2'$ in the second equation and solve for u_2', giving

$$u_2' = \frac{F(z)}{-(y_2/y_1)y_1' + y_2'} = \frac{F(z)y_1}{W(y_1, y_2)} \tag{13}$$

where $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is the *Wronskian determinant*. The result is that u_2 and u_1 are given by

$$u_2(z) = \int^z \frac{F(s)y_1(s)}{W(y_1(s), y_2(s))}\, ds, \quad u_1(z) = -\int^z \frac{F(s)y_2(s)}{W(y_1(s), y_2(s))}\, ds \tag{14}$$

and a particular solution for a second order inhomogeneous ODE is in general

$$y_p(z) = y_2(z) \int^z \frac{F(s)y_1(s)}{W(y_1(s), y_2(s))}\, ds - y_1(z) \int^z \frac{F(s)y_2(s)}{W(y_1(s), y_2(s))}\, ds. \tag{15}$$

[1] To understand the reason for the form of solutions to the ODE with constant coefficients a bit more, note that the ODE says that our sought function should be a linear combination of its derivatives. It is in principle only (can you come up with any other example?) the exponential function that retains its shape after differentiation, so therefore a solution that is equal to a linear combination of its derivatives must be an exponential. For the double root, note that the derivative of ze^{mz} is a linear combination of e^{mz} and ze^{mz} and the requirement that the solution is a linear combination of its derivatives will still be fulfilled.

[2] This follows from the fact that y_1 and y_2 are linearly independent so that the Wronskian determinant is nonzero.

and we can see that we do not need to bother about the integration constants as they will lead to terms in y_p proportional to y_1 and y_2. Here we have $F(s) = 6se^{2s}$ and

$$W(y_1(s), y_z(s)) = e^{2s}(1+2s)e^{2s} - 2e^{2s}se^{2s} = e^{4s}. \tag{16}$$

We then find

$$\begin{aligned} y_p(z) &= ze^{2z} \int^z \frac{6se^{2s}e^{2s}}{e^{4s}}\, ds - e^{2z} \int^z \frac{6se^{2s}se^{2s}}{e^{4s}}\, ds \\ &= 6ze^{2z} \int^z s\, ds - 6e^{2z} \int^z s^2\, ds \\ &= z^3 e^{2z} \end{aligned} \tag{17}$$

In the end, the general solution to our ODE is

$$y(z) = C_1 e^{2z} + C_2 z e^{2z} + z^3 e^{2z} \tag{18}$$

or expressed in terms of x

$$y(x) = C_1 x^2 + C_2 x^2 \ln x + x^2 \ln^3 x \tag{19}$$

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2. **P3, exam 2011-01-05.**

   (a) Use the Frobenius method to find one solution to the differential equation (note: it is a simple closed form): $x(x+1)\, y'' - (x-1)\, y' + y = 0$.

   (b) The Fuchsian conditions state that for the general second-order differential equation $y'' + f(x)y' + g(x)y = 0$ to give two independent solutions with the method of Frobenius it is necessary and sufficient that $xf(x)$ and $x^2 g(x)$ are expandable in convergent power series $\sum_{n=0}^{\infty} a_n x^n$. Show that the Fuchsian conditions hold for the equation in (a) above.

   (c) When the Fuchsian conditions hold,the general solution by the method of Frobenius will be either (i) two Frobenius series or (ii) one solution which is a Frobenius series $S_1(x)$ and a second solution $S_1(x)\ln x + S_2(x)$ where $S_2(x)$ is a second Frobenius series. Use case (ii) to find a second solution to the equation in (a) above.

   Solution.

   (a) In the Frobenius method we assume a series solution of form $y = \sum_{j=0}^{\infty} a_j x^{s+j}$, $a_0 \neq 0$, and insert this into the ODE and work out the solution from there. From Fuchs' theorem we are guaranteed that this will give as at least one of the two solutions to the second order ODE as long as we are expanding about a point that is at worst a regular singularity. With the series ansatz we find

$$\begin{aligned} x(x+1)y'' &= \sum_{j=0}^{\infty} \Big[a_j(s+j)(s+j-1)x^{s+j} + a_j(s+j)(s+j-1)x^{s+j-1} \Big] \\ &= a_0 s(s-1)x^{s-1} + \sum_{j=1}^{\infty} \Big[a_{j-1}(s+j-1)(s+j-2) \\ &\qquad + a_j(s+j)(s+j-1) \Big] x^{s+j-1} \qquad (20) \\ -(x-1)y' &= \sum_{j=0}^{\infty} \left(-a_j(s+j)\right) x^{s+j} + \sum_{j=0}^{\infty} a_j(s+j)x^{s+j-1} \\ &= a_0 s x^{s-1} + \sum_{j=1}^{\infty} \Big[-a_{j-1}(s+j-1) + a_j(s+j) \Big] x^{s+j-1} \qquad (21) \\ y &= \sum_{j=0}^{\infty} a_j x^{s+j} = \sum_{j=1} a_{j-1} x^{s+j-1} \qquad (22) \end{aligned}$$

Inserting this into the ODE yields

$$\begin{aligned} a_0(s(s-1)+s)x^{s-1} + \sum_{j=1}^{\infty} \Big[(s+j-1)\left((s+j-2)a_{j-1} + (s+j)a_j\right) \\ - a_{j-1}(s+j-1) + a_j(s+j) + a_{j-1} \Big] x^{s+j-1} = 0 \qquad (23) \end{aligned}$$

and the coefficients must vanish for each power of x separately. The lowest power of x is x^{s-1}, requiring this coefficient to be zero results in the indicial equation:

$$a_0(s(s-1)+s) = a_0 s^2 = 0 \quad \Leftrightarrow \quad s = 0 \qquad (24)$$

We have a double root to the indicial equation, so we can only get one series solution in this way. We will look at the other solution, which contains a logarithmic term, later. For $j \geq 1$ we get a recursion relation:

$$\begin{aligned} &(s+j-1)\left((s+j-2)a_{j-1} + (s+j)a_j\right) - a_{j-1}(s+j-1) + a_j(s+j) + a_{j-1} = 0 \\ \Leftrightarrow \quad & a_j = -a_{j-1}\frac{(j+s-2)^2}{(j+s)^2} \qquad (25) \end{aligned}$$

which with $s = 0$ becomes

$$a_j = -a_{j-1}\frac{(j-2)^2}{j^2}, \quad j = 1, 2, \ldots \qquad (26)$$

The series coefficients become

$$j = 1: \quad a_1 = -a_0 \tag{27}$$
$$j = 2: \quad a_2 = -a_1 \cdot 0 \quad \Rightarrow \quad a_2 = a_3 = a_4 = \ldots = 0 \tag{28}$$

so the first solution becomes rather simple, it is

$$y_1(x) = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1\ x = a_0\ (1 - x) \tag{29}$$

(b) The Fuchsian condition guarantees two solutions to our ODE if $xf(x)$ and $x^2 g(x)$ are analytic around the point which we are expanding, i.e. expandable as Taylor series, where the ODE is written as

$$y'' + f(x)y' + g(x)y = 0 \tag{30}$$

so that we have in our case

$$f(x) = \frac{1-x}{x(1+x)}, \quad g(x) = \frac{1}{x(1+x)}. \tag{31}$$

We will now check that the Fuchsian condition holds. Using the binomial series expansion[3] we get

$$xf(x) = \frac{1-x}{1+x} = (1-x)\sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1 \tag{32}$$
$$x^2 g(x) = \frac{x}{1+x} = x\sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1 \tag{33}$$

So the Fuchsian condition holds for $|x| < 1$, when the series converges.

(c) We now consider the second solution. The second solution is of the form

$$y_2(x) = y_1(x) \ln x + \sum_{j=0} b_j x^j. \tag{34}$$

To determine the behaviour of the b_j we insert y_2 into the ODE and simplify, this will give recursive relations between the b_j and the a_j. The derivatives

[3]The binomial expansion, convergent for $|x| < 1$, says that $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ and for $\alpha = -1$ we have

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n.$$

become

$$y_2' = y_1' \ln x + \frac{y_1}{x} + \sum_{j=0}^{\infty} b_j j x^{j-1} \tag{35}$$

$$y_2'' = y_1'' \ln x + \frac{y_1'}{x} + \frac{y_1'}{x} - \frac{y_1}{x^2} + \sum_{j=0}^{\infty} b_j j(j-1) x^{j-2} \tag{36}$$

Inserting this into the ODE we find

$$\underbrace{\mathcal{L}[y_1]}_{=0} \ln x + 2(x+1)y_1' \underbrace{- \frac{x+1}{x} y_1 - \frac{x-1}{x} y_1}_{=-2y_1} + \mathcal{L}\left[\sum_j b_j x^j\right] = 0$$

$$\Rightarrow \quad 2(x+1)y_1' - 2y_1 + \mathcal{L}\left[\sum_j b_j x^j\right] = 0 \tag{37}$$

where the first term in the first expression is zero since y_1 is a solution to the ODE so that $\mathcal{L}[y_1] = 0$. Operating with $\mathcal{L}$ on the series gives

$$\begin{aligned} \mathcal{L}\left[\sum_j b_j x^j\right] &= \sum_j b_j j(j-1) x^j + \sum_j b_j j(j-1) x^{j-1} \\ &\quad - \sum_j b_j j x^j + \sum_j b_j j x^{j-1} + \sum_j b_j x^j \\ &= \sum_j b_j j^2 x^{j-1} + \sum_j b_j (j-1)^2 x^j \end{aligned} \tag{38}$$

where all sums go from $j = 0$. Using the form for y_1 and $y_1' = -a_0$ we finally get from Eq. (37) the equation

$$4a_0 = \sum_{j=0}^{\infty} b_j j^2 x^{j-1} + \sum_{j=0}^{\infty} b_j (j-1)^2 x^j \tag{39}$$

This equation must hold power for power in x. We start with the lowest power of x and continue upwards. We find[4]

$$x^{-1}: \quad 0 = b_0 \cdot 0 \quad \Rightarrow \quad b_0 \text{ undetermined} \tag{40}$$

$$x^0: \quad 4a_0 = b_1 \cdot 1^2 + b_0(0-1)^2 \quad \Rightarrow \quad b_1 = 4a_0 - b_0 \tag{41}$$

For x^j with $j \geq 1$ we get a recursion relation for the b_j:

$$x^j,\ j \geq 1: \quad 0 = b_{j+1}(j+1)^2 + b_j(j-1)^2 \quad \Rightarrow \quad b_{j+1} = -b_j \left(\frac{j-1}{j+1}\right)^2 \tag{42}$$

[4]We have no parity arguments to put $b_0 = 0$ in this case, $\mathcal{L}$ has no definite parity.

and we see that the series terminates after $j = 1$ since

$$b_2 = -b_1 \cdot 0 \quad \Rightarrow \quad b_2 = b_3 = b_4 = \ldots = 0 \tag{43}$$

and therefore the second solution is given by

$$y_2(x) = y_1(x) \ln x + \sum_{j=0}^{\infty} b_j x^j = a_0(1-x) \ln x + b_0 + (4a_0 - b_0)x \tag{44}$$

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3. **P8.2.7.** $T_0(x) = 1$ and $V_1(x) = (1-x^2)^{1/2}$ are solutions of the Chebyshev ODE corresponding to different eigenvalues. Explain, in terms of the boundary conditions, why these functions are not orthogonal on the range $(-1, 1)$ with the appropriate weighting function.

   Solution. In *Sturm-Liouville theory* (SL theory) we consider self-adjoint (hermitian) differential operators on some interval (a, b) and their eigenvalues. Self-adjoint differential operators are of great interest since they have orthogonal eigenfunctions with real eigenvalues. The Chebyshev ODE is

$$(1-x^2)y'' - xy' + n^2 y = 0 \quad \Leftrightarrow \quad \mathcal{L}[y] = \lambda y. \tag{45}$$

   This can be put in self-adjoint form by multiplying with the weight function

$$w(x) = (1-x^2)^{-1/2} \tag{46}$$

   which means that its solutions (eigenfunctions) are orthogonal with the weight $w(x)$ on the interval (a, b) *if the boundary terms vanish.* In self-adjoint form the ODE is

$$\frac{d}{dx}\left[\bar{p}_0(x)\frac{dy}{dx}\right] + w(x)n^2 y = 0, \quad \bar{p}_0(x) = (1-x^2)^{1/2} \tag{47}$$

   If $u(x)$ and $v(x)$ are eigenfunctions of $\mathcal{L}$ we have that (see p. 385-386 in AWH)

$$\int_a^b w(x) v^* (\mathcal{L}u)\, dx = \Big[v^* \bar{p}_0 u' - (v^*)' \bar{p}_0 u\Big]_a^b + \int_a^b w(x)(\mathcal{L}v)^* u\, dx \tag{48}$$

   and the condition that $w\mathcal{L}$ is self-adjoint and hence u, v orthogonal when the inner product is weighted with $w(x)$ therefore requires that

$$\Big[v^* \bar{p}_0 u' - (v^*)' \bar{p}_0 u\Big]_a^b = 0 \tag{49}$$

   i.e. orthogonality condition *depends on the boundary conditions.*

   Here we have specifically the eigenfunctions

$$T_0 = 1, \quad V_1(x) = (1-x^2)^{1/2} \quad \Rightarrow \quad T_0' = 0, \quad V_1'(x) = -x(1-x^2)^{-1/2}. \tag{50}$$

These correspond to different eigenvalues and should therefore be orthogonal with the weight $w(x) = (1-x^2)^{-1/2}$. However, the boundary term does not vanish, we have with $v = T_0$, $u = V_1$, $\bar{p}_0 = (1-x^2)^{1/2}$ and $a = -1$, $b = 1$,

$$\Big[1(1-x^2)^{1/2}(-x)(1-x^2)^{-1/2} - 0\Big]_{-1}^{1} = \Big[-x\Big]_{-1}^{1} = -2 \neq 0 \tag{51}$$

The point is that V_1' is singular at the boundary. To ensure orthogonality we should specify boundary conditions such that our solutions and derivatives of them are bounded in the whole interval we are considering (on another interval than $(-1, 1)$ the V_n type of solutions may be fine).

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4. **P5, exam 2008-06-17.** An otherwise completely insulated rod of length L, has its endpoints connected to a heat bath which is kept, at all times, at zero temperature. In other words, the temperature at the endpoints does not change with time. At time $t = 0$ the temperature distribution in the rod is given by

$$u(x,0) = \begin{cases} \dfrac{2T_0}{L}\,x, & 0 \leqslant x \leqslant L/2 \\ \\ \dfrac{2T_0}{L}\,(L-x), & L/2 \leqslant x \leqslant L \end{cases}. \tag{52}$$

Determine the temperature distribution $u(x,t)$ for subsequent times.

Solution. The temperature distribution as a function of time and position $u(x,t)$ is in the case of a one-dimensional rod in the x-direction governed by the *partial* differential equation (PDE)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial u}{\partial t}, \tag{53}$$

where the constant $a \in \mathbb{R}$ measures the heat conductivity of the rod. To solve the PDE we assume that the solution can be written as a product of two functions where one is only a function of x and the other only a function of time t:

$$u(x,t) = X(x)T(t) \tag{54}$$

Inserting this in the PDE and dividing by $X(x)T(t)$ yields

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{aT(t)}\frac{dT}{dt} \equiv -k^2 \tag{55}$$

The right-hand side is only a function of t, whereas the left-hand side is only a function of x. When we vary t, nothing happens on the left-hand side and when we vary x nothing happens with the right-hand side, i.e. both sides must be equal to the same constant, which we call $-k^2$. We then get two *ordinary* differential equations, one for $X(x)$ and one for $T(t)$—we have *separated variables*:

$$\begin{cases} \dfrac{d^2X}{dx^2} + k^2 X = 0 \\ \\ \dfrac{dT}{dt} + ak^2 T = 0 \end{cases} \tag{56}$$

These are both ODE:s with constant coefficients and so have exponential solutions. The equation for t has solution (we put all constants in the spatial part of the solution)

$$T(t) = e^{-ak^2 t} \tag{57}$$

and from a physical point of view we see that we need $k^2 > 0$, i.e. k real, to get solutions that decay with time and go to zero for large t (there is no heat source in this problem, a temperature rising exponentially with time or oscillating indefinitely is unphysical). The spatial solutions are $e^{\pm ikx}$ and we write these here in terms of sines and cosines,

$$X(x) = A \sin kx + B \cos kx. \tag{58}$$

The boundary conditions say that the temperature is kept at all times at zero temperature, i.e. $u(0,t) = u(L,t) = 0, \forall t$. This requires

$$X(0) = X(L) = 0 \quad \Rightarrow \quad B = 0, \quad k = \frac{n\pi}{L} \equiv k_n,\ n = 0, 1, 2, \ldots \tag{59}$$

with all constants A_n (one for each n) still undetermined[5]. The general solution is obtained by summing all the solutions for every value of n,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-an^2\pi^2 t/L^2} \tag{60}$$

and we will now use the specified initial condition to determine the constants A_n,

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin k_n x. \tag{61}$$

We do this using the orthogonality of the $\sin k_n x$ functions over the interval from $x = 0$ to $x = L$,

$$\int_0^L \sin k_m x \sin k_n x \; dx = \frac{L}{2}\delta_{mn}. \tag{62}$$

[5] Assuming $(\partial u/\partial t)(0,t) = (\partial u/\partial t)(L,t) = 0$ gives the same condition since it just leads to multiplication of $X(0)$ and $X(L)$ by a constant $(-ak^2)$.

We integrate both sides in Eq. (61) over x from 0 to L together with $\sin k_m x$ and multiply with $(2/L)$ for correct normalisation:

$$\frac{2}{L}\int_0^L u(x,0)\sin k_m x\ dx = \frac{2}{L}\sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \sin k_n x \sin k_m x\ dx}_{=(L/2)\delta_{nm}}$$
$$= \sum_{n=0}^{\infty} A_n \delta_{nm}$$
$$= A_m \tag{63}$$

Inserting the explicit form of $u(x,0)$ given we find

$$A_n = \frac{2}{L}\frac{2T_0}{L}\left\{\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)\ dx\right\} \tag{64}$$

Using the integrals

$$\int_a^b \sin\left(\frac{n\pi x}{L}\right)\ dx = \left[\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_a^b \tag{65}$$

$$\int_a^b x\sin\left(\frac{n\pi x}{L}\right)\ dx = \left[x\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_a^b - \left(-\frac{L}{n\pi}\right)\int_a^b 1\cdot\cos\left(\frac{n\pi x}{L}\right)\ dx$$
$$= \left(-\frac{L}{n\pi}\right)\left(\left[x\cos\left(\frac{n\pi x}{L}\right)\right]_a^b - \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_a^b\right) \tag{66}$$

the A_n become

$$\begin{aligned}
A_n = \frac{4T_0}{L^2}\left(-\frac{L}{n\pi}\right)\Bigg\{ & \left[x\cos\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} - \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} \\
& + L\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} \\
& - \left(\left[x\cos\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L} - \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{L/2}^{L}\right)\Bigg\} \\
= -\frac{4T_0}{Ln\pi}\Bigg\{ & \frac{L}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\
& + L\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right) \\
& - \left(L\cos(n\pi) - \frac{L}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi}\sin(n\pi) + \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\Bigg\} \\
= -\frac{4T_0}{Ln\pi} & \left\{-\frac{2L}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right\} \\
= \frac{8T_0}{n^2\pi^2} & \sin\left(\frac{n\pi}{2}\right) \\
= & \begin{cases} 0, & n \text{ even} \\ \frac{8T_0}{n^2\pi^2}(-1)^{\frac{n+1}{2}+1}, & n \text{ odd} \end{cases}
\end{aligned} \tag{67}$$

The final solution for $u(x,t)$ is then

$$\begin{aligned}
u(x,t) &= \frac{8T_0}{\pi^2}\sum_{n \text{ odd}} \frac{(-1)^{\frac{n+1}{2}+1}}{n^2}\sin\left(\frac{n\pi x}{L}\right)e^{-an^2\pi^2 t/L^2} \\
&= \frac{8T_0}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^{m+2}}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)e^{-a(2m+1)^2\pi^2 t/L^2}
\end{aligned} \tag{68}$$